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Partial Differences of Vectors in Polar and Spherical Coordinates

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Shuman (1970) illustrated certain truncation errors associated with common difference estimates of derivatives in spherical coordinates, such as for divergence, D:

$$\overline{D} = \frac{\overline{u}_{\lambda}^{\phi}}{r \cos \phi_o} + \frac{\overline{v}_{\phi}^{\lambda}}{r} - \frac{\overline{v}^{\lambda\phi} \sin \phi_o}{r \cos \phi_o}$$

where ϕ_o is latitude at the center of a grid box. I showed these errors to be large in polar regions in grids in which the distance between points on a latitude circle are kept constant. In a regular latitude-longitude grid I showed these to be reduced by up to two orders of magnitude. For large-scale systems an error remained, however, barely less than the order of magnitude of vertically mass-averaged divergence in the atmosphere ($\sim 10^{-7}$ sec^{-1}).

Shuman (1976) later showed that the form given above does not conserve mass, and I showed how to correct it. These were studies of two different kinds of error, the first due to imperfect estimates of derivatives of vectors, and the second due to imperfect conservation of mass.

Howermale (1976) also dealt with the secondd kind of error. He emphasized ways to design difference systems at the pole itself.

In this paper, I will show how to eliminate both kinds of errors. The resulting correction factors are tiny in regular latitude-longitude grids, and can be significant only if they feed back upon themselves.

First, consider the problem in polar coordinates on a plane. Figure 1 shows the coordinate system and a grid box. In the hydrodynamic equations I first note that wherever a derivative of a velocity component with respect to α appears, a second term is always associated with it, thus:

$$\frac{\partial u}{\partial \alpha} - v$$

$$\frac{\partial v}{\partial \alpha} + u$$

The second terms in each expression arise from differentiation of a vector:

$$\frac{\partial \vec{V}}{\partial \alpha} = \left(\frac{\partial u}{\partial \alpha} - v\right)i + \left(\frac{\partial v}{\partial \alpha} + u\right)j$$

For convenience, and with some artistic license, I say

$$\frac{\partial \vec{V}}{\rho\,\partial\alpha} \cdot i = \frac{\partial u}{\rho\,\partial\alpha} - \frac{v}{\rho} \cong \frac{1}{A} \int \frac{\partial U}{\partial x}\, dA$$

$$\frac{\partial \vec{V}}{\partial\rho} \cdot j = - \frac{\partial v}{\partial\rho} \cong - \frac{1}{A} \int \frac{\partial V}{\partial y}\, dA$$

I now choose U and V to be functions of space, such that they are each linear along all straight lines α = constant and all straight lines y = constant. At constant y,

$$U = \overline{U}^{x} + U_x\, x$$

But

$$x = -y \tan\alpha$$

$$\overline{U}^{x} = \overline{U}^{\alpha}$$

$$U_x = \frac{U_\alpha}{x_\alpha}$$

$$x_\alpha = \frac{-2y \tan \frac{1}{2}\Delta\alpha}{\Delta\alpha} = -y\,\frac{S}{C}$$

where

$$S = \frac{\sin \frac{1}{2}\Delta\alpha}{\frac{1}{2}\Delta\alpha}$$

$$C = \cos \tfrac{1}{2}\Delta\alpha$$

Therefore, at constant y,

$$U = \overline{U}^{\alpha} + \frac{C}{S}\, U_\alpha \tan\alpha \qquad (1)$$

Now

$$\overline{U}^{\alpha} = \overline{U}^{\alpha\rho} + \overline{U}^{\alpha}_{\rho}\, (\overline{\rho}^{\alpha} - \rho_o)$$

But

$$\rho = - \frac{y}{\cos\alpha}$$

$$\overline{\rho}^{\alpha} = - \frac{y}{C}$$

$$\rho_o = - \frac{y_9}{C}$$

Therefore,

$$\overline{U}^{\alpha} = \overline{U}^{\alpha\rho} - \frac{1}{C}\, \overline{U}^{\alpha}_{\rho}\, (y - y_9) \qquad (2a)$$

Similarly,

$$U_\alpha = \overline{U}^{\rho}_{\alpha} - \frac{1}{C}\, U_{\alpha\rho}\, (y - y_9) \qquad (2b)$$

Substitution from (2) into (1) yields

$$U = \overline{U}^{\alpha\rho} + \frac{C}{S}\,\overline{U}^{\rho}_{\alpha} \tan\alpha - \frac{1}{C}\,\overline{U}^{\alpha}_{\rho}\,(y-y_9) - \frac{1}{S}\,U_{\alpha\rho}\,(y-y_9)\tan\alpha \tag{3a}$$

Similarly for V,

$$V = \overline{V}^{\alpha\rho} + \frac{C}{S}\,\overline{V}^{\rho}_{\alpha} \tan\alpha - \frac{1}{C}\,\overline{V}^{\alpha}_{\rho}\,(y-y_9) - \frac{1}{S}\,V_{\alpha\rho}\,(y-y_9)\tan\alpha \tag{3b}$$

Equations (3) show that my assumption about linear variation of $\vec{V}$ along the two sets of straight lines is neither an over-statement nor an under-statement. Such a variation will yield the given sets of values at the four bounding grid points of the box.

Now

$$\frac{1}{A}\int \frac{\partial U}{\partial x}\,dA = \frac{1}{A}\int_{y_4}^{y_8}\int_{x_2}^{x_6} \frac{\partial U}{\partial x}\,dx\,dy$$

$$= \frac{\Delta\alpha}{A}\int_{y_4}^{y_8} U_\alpha\,dy$$

$$= \frac{\Delta\alpha}{A}\int_{y_4}^{y_8} [\overline{U}^{\rho}_{\alpha} - \frac{1}{C}\,U_{\alpha\rho}\,(y-y_9)]dy$$

$$= -\frac{\Delta\alpha\ \Delta\rho}{A}\,\overline{y}^{\alpha}_{\rho}\,\overline{U}^{\rho}_{\alpha}$$

But $\quad y = -\rho\cos\alpha$

$$\overline{y}^{\alpha}_{\rho} = -C$$

Therefore

$$\frac{1}{A}\int \frac{\partial U}{\partial x}\,dA = +\frac{C\ \Delta\alpha\ \Delta\rho}{A}\,\overline{U}^{\rho}_{\alpha}$$

The area of the grid box is

$$A = \int_{y_4}^{y_8}\int_{x_2}^{x_6} dx\,dy = \Delta\alpha\int_{y_4}^{y_8} x_\alpha\,dy = -\frac{S}{C}\,\Delta\alpha\int_{y_4}^{y_8} y\,dy$$

$$= +\frac{S}{C}\,\Delta\alpha\ \Delta\rho\,\frac{[(\overline{y}^{\alpha})^2]_\rho}{2} = +\frac{S}{C}\,\Delta\alpha\ \Delta\rho\ \overline{y}^{\rho\alpha}\ \overline{y}^{\alpha}_{\rho}$$

But

$$y = -\rho \cos \alpha$$

$$\bar{y}^{\alpha\rho} = -\rho_o \ C$$

and therefore

$$A = SC \ \rho_o \ \Delta\alpha \ \Delta\rho$$

Therefore,

$$\frac{1}{A} \int \frac{\partial U}{\partial x} \, dA = \frac{1}{S} \frac{\bar{U}^{\rho}_{\alpha}}{\rho_o}$$

Now,

$$\tan \alpha = -\frac{x}{y}$$

$$\frac{\partial \tan \alpha}{\partial y} = \frac{x}{y^2}$$

Therefore, differentiation of (3b) yields

$$\frac{\partial V}{\partial y} = \frac{C}{S} \bar{V}^{\rho}_{\alpha} \frac{x}{y^2} - \frac{1}{C} \bar{V}^{\alpha}_{\rho} + \frac{1}{S} V_{\alpha\rho} \ y_9 \frac{x}{y^2}$$

$$= -\frac{1}{C} \bar{V}^{\alpha}_{\rho} + \frac{1}{S} \left(C \ \bar{U}^{\rho}_{\alpha} + V_{\alpha\rho} \ y_9 \right) \frac{x}{y^2}$$

and then, because the x-integration is zero by symmetry,

$$\frac{1}{A} \int \frac{\partial V}{\partial y} \, dA = -\frac{1}{C} \bar{V}^{\alpha}_{\rho}$$

In summary,

$$\frac{1}{A} \int \frac{\partial U}{\partial x} \, dA = \frac{\bar{U}^{\rho}_{\alpha}}{S \rho_o} \quad (4a)$$

$$\frac{1}{A} \int \frac{\partial V}{\partial y} \, dA = -\frac{\bar{V}^{\alpha}_{\rho}}{C} \quad (4b)$$

Now,

$$x = \rho \sin \alpha$$

$$y = -\rho \cos \alpha$$

Therefore,

$$U = u \cos \alpha - v \sin \alpha \quad (5a)$$

$$V = u \sin \alpha + v \cos \alpha \quad (5b)$$

and

$$\bar{U}^{\rho}_{\alpha} = \bar{u}^{\rho}_{\alpha}\ C - \bar{v}^{\alpha\rho}\ S$$

$$\bar{V}^{\alpha}_{\rho} = \left(\frac{\Delta\alpha}{2}\right)^2 u_{\alpha\rho}\ S + \bar{v}^{\alpha}_{\rho}\ C$$

and therefore,

$$\frac{\partial u}{\rho\,\partial\alpha} - \frac{v}{\rho} \cong \frac{1}{A}\int \frac{\partial U}{\partial x}\ dA = \frac{C}{S}\,\frac{\bar{u}^{\rho}_{\alpha}}{\rho_o} - \frac{\bar{v}^{\alpha\rho}}{\rho_o} \tag{6a}$$

$$\frac{\partial v}{\partial\rho} \cong \frac{1}{A}\int \frac{\partial V}{\partial y}\ dA = -\ \bar{v}^{\alpha}_{\rho} - \frac{1-C^2}{SC}\ u_{\alpha\rho} \tag{6b}$$

In the latter, I have used the identity,

$$\left(\frac{\Delta\alpha}{2}\right)^2 \equiv \frac{1-C^2}{S^2}$$

There is a conceptually straightforward analogous derivation on the surface of a sphere, but it involves messy integrals. I will outline here its principles, but will adopt a simpler approach. In brief, if X, Y, and Z are cartesian coordinates as shown in figure 2, and λ is longitude, ϕ latitude, and r radial distance from the center of the earth, then λ and $-\phi$ are analogous to α and ρ on the plane. The analogues to x and y on the plane are gotten by rotating the coordinate system 90° about the X-axis to obtain new sets of coordinates, X, Y', Z' and λ', ϕ', r, as shown. If the X, Z-plane cuts through the middle of a grid "box," the coordinate ϕ' is analogous to x on the plane, and $-\lambda'$ is analogous to y. The relations among the various coordinates are given by

$$X = X' = r\cos\phi\cos\lambda = r\cos\phi'\cos\lambda' \tag{7a}$$
$$Y = Z' = r\cos\phi\sin\lambda = r\sin\phi' \tag{7b}$$
$$Z = -Y' = r\sin\phi = -r\cos\phi'\sin\lambda' \tag{7c}$$

and

$r\cos\phi\,\dot{\lambda}$	is analogous to u on the plane
$r\,\dot{\phi}$	" " " v " " "
$r\,\dot{\phi}'$	" " " U " " "
$-r\cos\phi'\dot{\lambda}'$	" " " V " " "
$\dfrac{\partial(r\,\dot{\phi}')}{r\,\partial\phi'}$	" " " $\dfrac{\partial U}{\partial x}$ " " "
$\dfrac{\partial(r\cos\phi'\dot{\lambda}')}{r\cos\phi'\,\partial\lambda'} - \dfrac{(r\,\dot{\phi}')\sin\phi'}{r\cos\phi'}$	" " " $\dfrac{\partial V}{\partial y}$ " " "

A grid "box" on the spherical surface is bounded by the curves, λ = constant and λ' = constant, all being great circles and analogous to the straight lines, α = constant and y = constant on the plane,

A difference system is a set of approximations, and simplifications are therefore justified. I will simplify the problem through conformal mapping. Horizontal divergence expressed in cartesian coordinates on a conformal projection is

$$D = m^2 \left[\frac{\partial}{\partial x}\left(\frac{U}{m}\right) + \frac{\partial}{\partial y}\left(\frac{V}{m}\right)\right] \tag{8}$$

where m is the map scale factor, i.e., the ratio of a mapped small distance to the distance true on the earth; and U and V are the x,y- components of velocity true on the earth. I will choose the mapping function so that at the center of the grix box, x increases eastward and y northward. Equation (8) then suggests to me the approximations

$$\frac{\partial u}{r \cos \phi\, \partial\lambda} - \frac{v \sin \phi}{r \cos \phi} \cong \frac{1}{A} \int m^2 \frac{\partial}{\partial x} \left(\frac{U}{m}\right) dA \tag{9a}$$

$$\frac{\partial v}{r \partial\phi} \cong \frac{1}{A} \int m^2 \frac{\partial}{\partial y} \left(\frac{V}{m}\right) dA \tag{9b}$$

together with appropriate assumptions of linear variation of U/m and V/m with distance on the map. In (9) A is the area of the grid box true on the earth:

$$A = \int_{y_4}^{y_8} \int_{x_2}^{x_6} m^{-2} \, dx\, dy$$

I choose a Mercator projection true at $\phi' = \lambda = 0$, that is, the "equator" of the projection will be the central geographical meridian of the grid box. My choice has the advantage that a north-south column of grid boxes, geographical pole-to-pole, are all mapped on the same projection. The mapping functions, then, are

$$x = r \ln \left(\frac{1 + \sin \phi'}{\cos \phi'}\right) \tag{10a}$$

$$y = -r\lambda' \tag{10b}$$

$$m = \frac{1}{\cos \phi'} \tag{10c}$$

where λ', ϕ' are related to λ, ϕ by (7), and are shown in Figure 2.

Now, generally, the great circles bounding the box, i.e., $\lambda_{2,6}$ = constant and $\lambda'_{4,8}$ = constant, cannot all be straight lines on a conformal map. In the Mercator projection that I am adopting, the northern and southern boundaries are straight lines on the map, namely, $y_{4,8} = r\lambda'_{4,8}$ = constant. The eastern and western boundaries, however, are convex curves. This means I cannot use directly my results (4), that I derived for polar coordinates on the plane, to evaluate the integrals in (9).

Instead of attempting to carry out the quadratures in (9), I will approximate them, guided by the forms (4). Thus, I take

$$\frac{1}{A}\int m^2 \frac{\partial}{\partial x}\left(\frac{U}{m}\right) dA \simeq \frac{k_1}{r \cos \phi_o} \overline{\left(\frac{U}{m}\right)_{\lambda}}^{\phi} \tag{11a}$$

$$\frac{1}{A}\int m^2 \frac{\partial}{\partial y}\left(\frac{V}{m}\right) dA \simeq \frac{k_2}{r} \overline{\left(\frac{V}{m}\right)_{\phi}}^{\lambda} \tag{11b}$$

where k_1 and k_2 are functions of Δ alone, the interval of latitude and longitude between grid points, and will be chosen below by conservation arguments.

Now, the eastward and northward components of velocity are

$$u = r \cos \phi \, \dot{\lambda} \tag{12a}$$

$$v = r \, \dot{\phi} \tag{12b}$$

The x and y components of velocity true on the earth are

$$U = \frac{\dot{x}}{m} = r \, \dot{\phi}' \tag{13a}$$

$$V = \frac{\dot{y}}{m} = -r \cos \phi' \, \dot{\lambda}' \tag{13b}$$

which can easily be obtained by differentiating (10a) and (10b). Just as on the plane, there must be a relationship like (5) between U,V and u,v as defined by (13) and (12). The existence of such a relationship is necessitated by conformality, which is a sufficient condition for preservation of angles. I write (5) again, this time with U,V and u,v defined by (13) and (12):

$$U = u \cos \alpha - v \sin \alpha \tag{14a}$$

$$V = u \sin \alpha + v \cos \alpha \tag{14b}$$

I solve these for u,v in terms of U,V:

$$u = U \cos \alpha + V \sin \alpha \tag{15a}$$

$$v = -U \sin \alpha + V \cos \alpha \tag{15b}$$

Differentiating (7b):

$$u \cos \lambda - v \sin \phi \sin \lambda = U \cos \phi'$$

and comparing this with (14a), I find

$$\cos \alpha = m \cos \lambda \tag{16a}$$

$$\sin \alpha = m \sin \phi \sin \lambda \tag{16b}$$

I can get another set of expressions for α by differentiating (7c):

$$v \cos \phi = U \sin \phi' \sin \lambda' + V \cos \lambda'$$

Comparing this with (15b) I find

$$\cos \alpha = \frac{\cos \lambda'}{\cos \phi}$$

$$\sin \alpha = - \frac{\sin \phi' \sin \lambda'}{\cos \phi}$$

These two sets, of course, are equivalent, as may easily be shown by manipulations using (7). Using (14) and (16) I now write

$$\frac{U}{m} = u \cos \lambda - v \sin \phi \sin \lambda \tag{17a}$$

$$\frac{V}{m} = u \sin \phi \sin \lambda + v \cos \lambda \tag{17b}$$

For convenience and economy in writing, I define

$$s = \frac{\sin \frac{1}{2}\Delta}{\frac{1}{2}\Delta}$$

$$c = \cos \tfrac{1}{2}\Delta$$

and note that

$$\frac{\Delta^2}{4} = \frac{1-c^2}{s^2}$$

$$(\cos \lambda)_\lambda = \overline{(\sin \lambda)}^\lambda = 0$$

$$\overline{(\cos \lambda)}^\lambda = c$$

$$(\sin \lambda)_\lambda = s$$

$$(ab)_z = \bar{a}^z b_z + a_z \bar{b}^z$$

$$\overline{(ab)}^z = \bar{a}^z \bar{b}^z + \frac{1-c^2}{s^2} a_z b_z$$

where a,b are any variables, and z is one coordinate of any pair of coordinates. The other coordinate of the pair is understood to be held constant in differencing and averaging.

I now perform the difference operations in (11) on (17):

$$\frac{1}{A} \int m^2 \frac{\partial}{\partial x} \left(\frac{U}{m}\right) dA \cong \frac{k_1}{r \cos \phi_o} \left[c\bar{u}^\phi_\lambda - s\overline{(\bar{v}^\lambda \sin \phi)}^\phi \right] \tag{18a}$$

$$\frac{1}{A} \int m^2 \frac{\partial}{\partial y} \left(\frac{V}{m}\right) dA \cong \frac{k_2}{r} \left[\frac{1-c^2}{s} (u_\lambda \sin \phi)_\phi + c\bar{v}^\lambda_\phi \right] \tag{18b}$$

When (18a) is added to (18b) an estimate, $\bar{D}$, for divergence is obtained. When that estimate $\bar{D}$ is weighted by the area of its grid box, and summed over an ensemble of adjacent boxes, the result should be a "line" sum around the boundary of the ensemble. In other words, the result should depend only on values of u and v on the boundary of the ensemble, and not at all on values interior to the ensemble. At least such a feature is desirable and can easily be designed into the estimates (18) by an appropriate choice of k_1 and k_2, as I shall show.

The area of a grid box bounded by two meridians and two latitude circles each separated by Δ is

$$sr^2 \Delta^2 \cos \phi_o$$

This is not precisely the area of my grid box, which is bounded on the north and south by great circles rather than latitude circles. It is a close approximation, however, in the spirit of difference approximations. In any case, the precise area of the grid box will not enter into the system that I am deriving. I will only use the dependence of the area on the central latitude.

Thus, after multiplying (18a) and (18b) by $(\cos \phi_o)$, I note that the first term of the right-hand member of (18a), when summed box-by-box west-to-east, will sum to the difference of $\bar{u}^\phi$ at the extrema of the east-west row. Similarly, because ϕ is constant east-west, the first term of (18b) will sum to values only at the extrema of an east-west row. That leaves the second terms of (18a) and (18b) to be considered. I will choose k_1 and k_2 so that the second terms sum to values only at the extrema of a north-south column of boxes. Now, consider them added together and multiplied by $(r \cos \phi_o)$:

$$\begin{aligned} & -k_1 s \, \overline{(\bar{v}^\lambda \sin \phi)}^\phi + k_2 c \bar{v}^\lambda_\phi \cos \phi_o \\ &= -k_1 s \left(c \bar{v}^{\lambda\phi} \sin \phi_o + \frac{1-c^2}{s} \bar{v}^\lambda_\phi \cos \phi_o\right) + k_2 c \bar{v}^\lambda_\phi \cos \phi_o \\ &= -k_1 sc \bar{v}^{\lambda\phi} \sin \phi_o + [-k_1(1-c^2) + k_2 c] \bar{v}^\lambda_\phi \cos \phi_o \qquad (19a) \end{aligned}$$

I have used here the following:

$$\overline{(\sin \phi)}^\phi = c \sin \phi_o$$

$$(\sin \phi)_\phi = s \cos \phi_o$$

Operations on $(\cos \phi)$ yield similar results:

$$\overline{(\cos \phi)}^\phi = c \cos \phi_o$$

$$(\cos \phi)_\phi = -s \sin \phi_o$$

According to Shuman (1976), the areal average, $\overline{D}$, of divergence over a grid "box" bounded by pairs of meridians and latitude circles is

$$\overline{D} = \frac{1}{sr \cos \phi_o} [\overline{u}^{\phi}_{\lambda} + (\overline{v}^{\lambda} \cos \phi)_{\phi}]$$

Consider the term involving v after multiplying by ($r \cos \phi_o$):

$$\frac{1}{s} (\overline{v}^{\lambda} \cos \phi)_{\phi} = -\overline{v}^{\lambda\phi} \sin \phi_o + \frac{c}{s} \overline{v}^{\lambda}_{\phi} \cos \phi_o \tag{19b}$$

This term plainly sums to values at the extrema of a north-south column. Further, it is evident that (19a) and (19b) are estimates of the same quantity if k_1 and k_2 are nearly unity. I will make (19a) and (19b) the same by writing

$$k_1 sc = 1$$

$$-k_1(1-c^2) + k_2 c = \frac{c}{s}$$

Thus,

$$k_1 = (sc)^{-1}$$

$$k_2 = (sc^2)^{-1}$$

and (9) and (18) become

$$\frac{\partial u}{r \cos \phi \, \partial\lambda} - \frac{v \sin \phi}{r \cos \phi} \cong \frac{1}{A} \int m^2 \frac{\partial}{\partial x} \left(\frac{U}{m}\right) dA$$

$$\cong \frac{1}{r \cos \phi_o} \left[\frac{1}{s} \overline{u}^{\phi}_{\lambda} - \frac{1}{c} \overline{(\overline{v}^{\lambda} \sin \phi)}^{\phi}\right] \tag{20a}$$

$$\frac{\partial v}{r\partial\phi} \cong \frac{1}{A} \int m^2 \frac{\partial}{\partial y} \left(\frac{V}{m}\right) dA \cong \frac{1}{rsc} \left[\overline{v}^{\lambda}_{\phi} + \frac{1-c^2}{sc} (u_{\lambda} \sin \phi)_{\phi}\right] \tag{20b}$$

The estimates for the ϕ-derivative of u and the λ-derivative of v may be obtained directly from (20a) and (20b) by substituting components of the vector $\vec{V}\times k$, where k is a unit vertical vector. Thus, wherever u appears above I substitute v, and wherever v appears I substitute $-u$. Then,

$$\frac{\partial v}{r \cos \phi \, \partial\lambda} + \frac{u \sin \phi}{r \cos \phi} \cong \frac{1}{A} \int m^2 \frac{\partial}{\partial x} \left(\frac{V}{m}\right) dA$$

$$\cong \frac{1}{r \cos \phi_o} \left[\frac{1}{s} \overline{v}^{\phi}_{\lambda} + \frac{1}{c} \overline{(\overline{u}^{\lambda} \sin \phi)}^{\phi}\right] \tag{20c}$$

$$\frac{\partial u}{r\partial\phi} \cong \frac{1}{A} \int m^2 \frac{\partial}{\partial y} \left(\frac{U}{m}\right) dA \cong \frac{1}{rsc} \left[\overline{u}^{\lambda}_{\phi} - \frac{1-c^2}{sc} (v_{\lambda} \sin \phi)_{\phi}\right] \tag{20d}$$

Use of the Corrected Difference Forms.

The finite difference systems (20) developed here eliminate entirely the errors discussed by Shuman (1970). One conclusion that may be drawn, therefore, is that my criticism of grids like Kurihara's (1965), in which distances between grid points are preserved, do not apply if finite difference systems are derived along the lines presented here.

In regular latitude-longitude grids commonly in use, the corrections arising from curvilinearity are typically small, less than 10^{-7} sec^{-1}. Such small magnitudes suggest that they are not important, except perhaps in their effect on vertically mass-averaged divergence, for its order of magnitude is typically 10^{-7} sec^{-1}. At any rate, the continuity equation is the first place I would look for significant effects.

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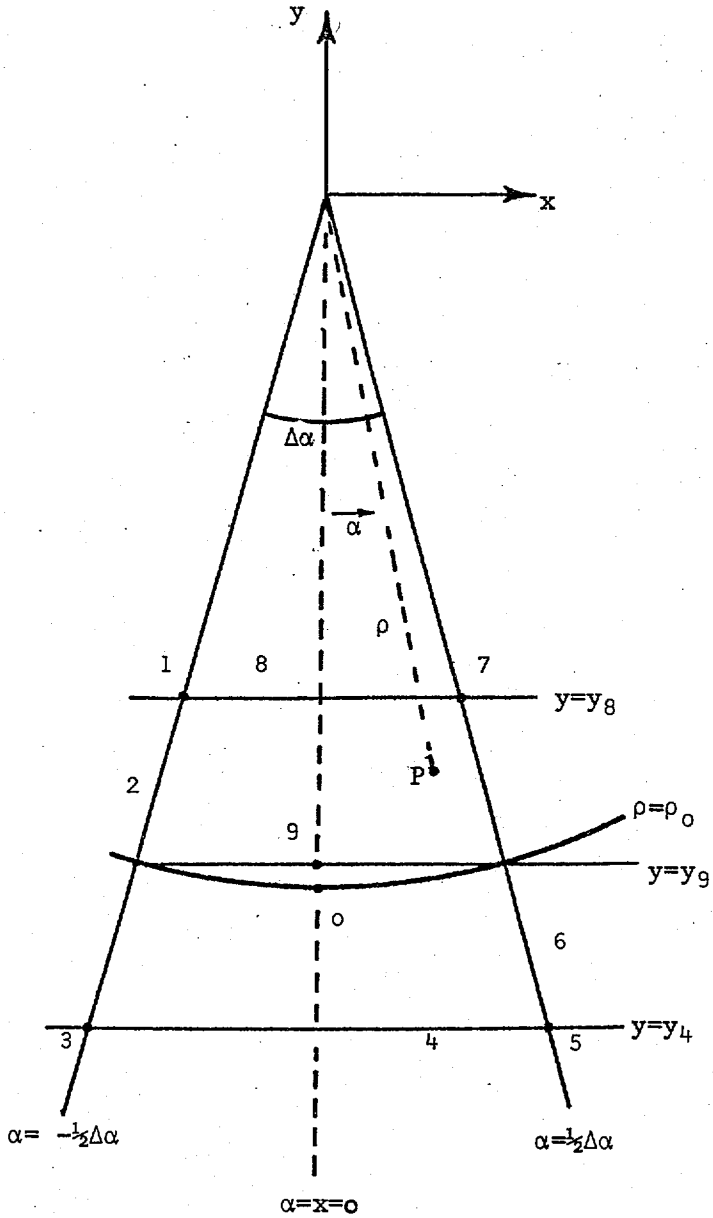


$$x = \rho \sin \alpha$$
$$y = -\rho \cos \alpha$$

$$u = \rho \dot{\alpha}$$
$$v = -\dot{\rho}$$

$$i = \rho \nabla \alpha$$
$$j = - \nabla \rho$$

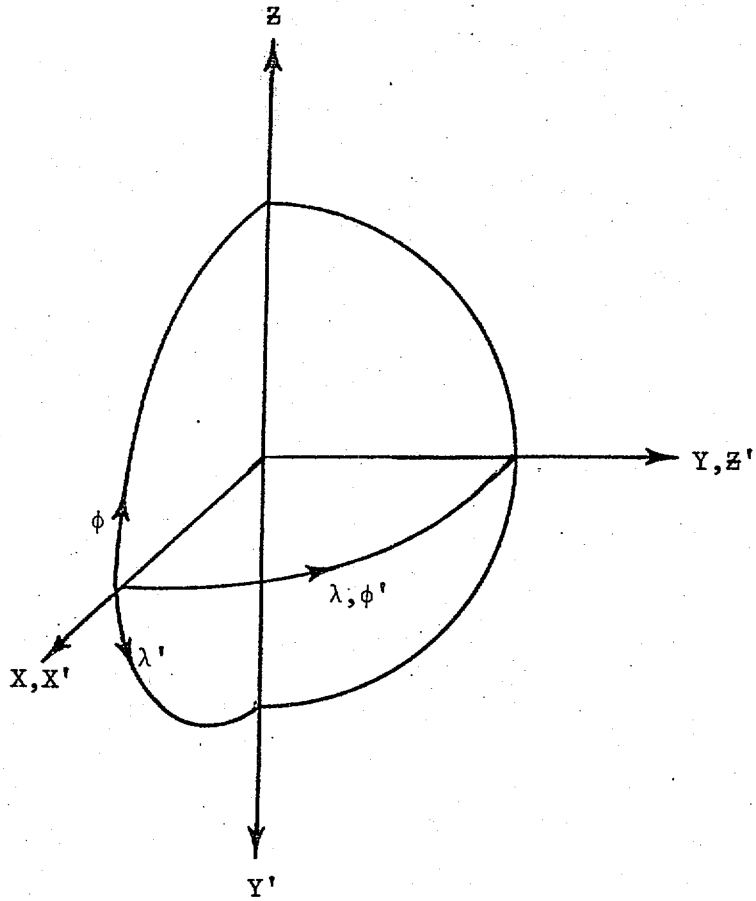
$$U = \dot{x}$$
$$V = \dot{y}$$

$$A = \int_{y_4}^{y_8} \int_{x_2}^{x_6} dx \; dy$$

$$\frac{1}{A} \int (\;) \, dA = \int_{y_4}^{y_8} \int_{x_2}^{x_6} (\;) \, dx \; dy$$

$$\Delta\alpha = \alpha_6 - \alpha_2$$
$$\Delta\rho = \rho_3 - \rho_1 = \rho_5 - \rho_7$$

FIGURE 1



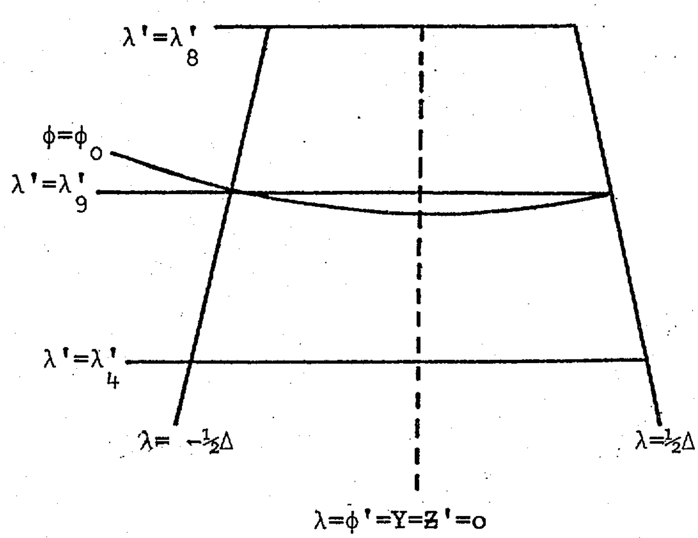


FIGURE 2