Computation of Mandatory Level Heights Given \( \sigma \) Layer PE Forecasts of Heights and Temperatures (A Revised Method)

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APRIL 1974
COMPUTATION OF MANDATORY LEVEL HEIGHTS
GIVEN $\sigma$ LAYER PE FORECASTS OF HEIGHTS AND TEMPERATURES
(A REVISED METHOD)

The basic procedure, which is not altered, is to take the hydrostatic equation in the form

$$\frac{\partial g\theta}{\partial \pi} = -c_p \theta \tag{1}$$

$$\pi = \frac{R}{C_p} \frac{P}{P_0} \quad P = 1000 \text{ centipascals} \tag{2}$$

and integrate it from a $\sigma$ surface of known $z$ and $\pi$ to a mandatory level of known $\pi$ (and therefore $\pi$) thus obtaining the value of $z$ on the mandatory level. In order to do this mathematically, it is necessary to make a statement relating $\theta$ and $\pi$ — what has been done (and will continue to be done) is to assume a linear variation of $\theta$ with $\pi$:

$$\theta = a + b(\pi - \pi_0) \tag{3}$$

$\pi_0$ is a particular known value of $\pi$ at which $\theta$ is known and thus $a = \theta_0$; $\theta_0 - \pi_0$ is a reference point to which to tie the linear variation; $b$ represents the slope $\Delta \theta/\Delta \pi$ of the $\theta-\pi$ line and requires known values of $\theta$ and $\pi$ at two levels for its specification. And therein lies the difficulty—what levels do we select from the forecast $\sigma$-coordinate information to specify the values of $a$ and $b$?

But first let us substitute (3) into (1) and do the integration from a $\sigma$ surface of known $z_\sigma$ to the desired pressure surface to obtain the wanted $z_p$. The result is

$$z_p = z_\sigma - \frac{c_p}{g} \left[ a(\pi_p - \pi_\sigma) + \frac{b}{2} (\pi_p - \pi_0)^2 - (\pi_\sigma - \pi_0)^2 \right] \tag{4}$$

where $\pi_p$ and $\pi_\sigma$ are the $\pi$ values at the known pressure and $\sigma$ surfaces respectively.

Returning to the question of how to specify $a$ and $b$ (and $\theta_0$, $\pi_0$), recall that the PE models (either 6- or 8-layer) forecast temperature and pressure difference in or across layers. From these, it is straightforward to compute height and pressure (and $\pi$) at $\sigma$ levels. This then constitutes our $\sigma$ coordinate information: $z$ and $\pi$ at $\sigma$-levels, and $\theta$ in the layers between. As it is necessary to assign a value of $\pi$ to the middle of the layer and attach our layer $\theta$ to that value, the mean value of the $\pi$'s at the adjacent levels is computed for this purpose. (See Tech. Proc. Bull. #11, 13 Feb. 1968)
Fig. 1 illustrates the situation (in the vicinity of the tropopause)

The solid lines are numbered $\sigma$ surfaces, the dashed line is the mandatory $p$ surface for which we want $z_p$.

The old procedure was, given a value of $p$ and $\pi_p$, to search out the layer mean $\pi$ values which bracket $\pi_p (\pi_3$ and $\pi_2$ in the particular situation of Fig. 1) then define, for the particular example,

$$b = \frac{\theta_3.5 - \theta_2.5}{\pi_3.5 - \pi_2.5}$$

$$a = \theta_2.5; \ (\pi_0 = \pi_2.5)$$

and integrate from the $\sigma$ level separating the layers ($z_\sigma = z_2$) to the pressure level. This integration could go up or down from $\sigma$ level 3 depending upon whether $p$ or $\pi_p$ was less or greater than $\pi_3$, but the integration would not go beyond the center ($\pi_3$) of either layer. If the $\pi_p$ were on the "other side" of the center of the layer, the integration would start from the next $\sigma$ level up or down.

The new procedure is, given the same value of $p$ and $\pi_p$, to search out the $\sigma$-level $\pi$ values which bracket $\pi_p$ ($\pi_3$ and $\pi_2$ in this case). At those levels then we further define values of $\theta$ obtained by interpolation of $\theta$, linear with $\pi$, from the adjacent layers. For example

$$\theta_2 = \frac{\theta_1.5(\pi_2.5 - \pi_2) + \theta_2.5(\pi_2 - \pi_1.5)}{\pi_2.5 - \pi_1.5}$$
Then the slope of the $\theta - \pi$ line is

\[ b = \frac{\theta_3 - \theta_2}{\pi_3 - \pi_2} \]

while

\[ a = \theta_{2.5} \quad (\pi_0 = \pi_{2.5}) \]

for the particular example. The integration then proceeds from $\theta_0 = z_3$ up to the desired pressure level. In this case, the integration could go beyond the middle of the layer but does not go past the next $\sigma$ level.

The only difference between the two procedures is in the specification of the slope term $b$. And indeed through most of the atmosphere (anywhere below level 3 or above 2 in Fig. 1) the old vs. new values give all but identical results for $b$ and thus for the calculated heights. They differ by no more than a meter or two as shown by tests.

However, it is in the layer under the tropopause, where a large change in the lapse rate takes place, that the old procedure causes difficulties. In effect, the old procedure causes (in Fig. 1) $z_p$ to be larger than the value calculated by the new. This is not difficult to see as the old procedure integrates along the $\theta_3$ to $\theta_{2.5}$ line while the new would integrate along a line parallel to the $\theta_3$ to $\theta_2$ line but drawn through $\theta_{2.5}$. The old method is thus warmer than the new, causing the difference in $z_p$.

Similarly, if the desired pressure level lay in the upper half of the same layer a similar thing happens but with an important reversal. The old method, being warmer, gives a lower value to $z_p$ (we are integrating down now instead of up) than does the new.

If, as is frequently the case, two mandatory levels fall in the above positions—one in the upper half, one in the lower half of the layer under the tropopause—the old procedure produces a mandatory layer thickness temperature considerably colder than the new. Frequently, this thickness temperature is actually substantially unstable (by as much as 20°K) with respect to the mandatory layer below while the new method thickness and its associated temperature gives no trouble at all. This is obviously a correction to an error.

Also, it is possible to show that if the $\pi$ level happened to coincide with $\pi_{2.5}$ you would get different values of $z_p$ depending upon whether you integrated down from level 2 or up from 3 using the old procedure but you would get the same correct value for $z_p$ using the new.

For these reasons, we are replacing the old procedure with the new in all NMC PE models—the 6L coarse mesh, the LFM and the 8L GLOBAL and Hemispheric models.