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OFFICE NOTE 93

On the Constraints Which Must Be Satisfied to Yield Vertically Interpolated Approximations of the Pressure Gradient in Sigma Coordinates

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1.0 Introduction

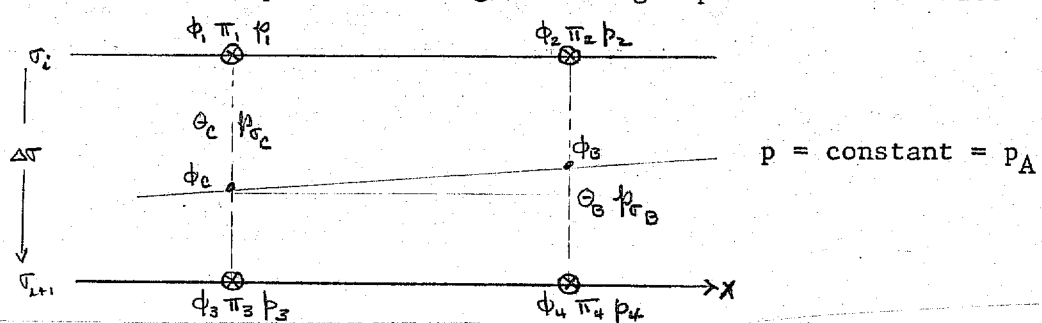
One aspect of the sigma coordinate formulation of the weather prediction equations which has been a source of difficulty is the estimation of the horizontal pressure gradient force.

In this note, certain numerical aspects of the approximation considered as an interpolation are discussed. As a general rule, numerical approximation by interpolation is more accurate and stable than an approximation involving extrapolation.

It is shown below that constraints exist for the "interpolational" character to be posited for approximations of the type used in NMC models. Two alternative schemes are proposed for approximation of the pressure gradient. One is subject to similar constraints as the present NMC method; the other scheme is not subject to such contraints since it always involves vertical interpolation.

2.0 The Interpolation Constraint

Consider x-σ plane: Let ⊗ denote gridpoints on σ-surface



The sloping line denotes an isobaric coordinate surface. ϕ_C and ϕ_B are the geopotentials at the intersection of the isobaric surface, p_A, with the vertical lines connecting σ-surface grid points. θ_C, θ_B, p_{σ_C} and p_{σ_B} are the values of the parameters in the layer between the σ surfaces.

They are defined by the relations

$$p_{\sigma_C} = \frac{p_3 - p_1}{\Delta\sigma} \qquad p_{\sigma_B} = \frac{p_4 - p_2}{\Delta\sigma}$$

$$\theta_C = -\frac{1}{c_p}\,\frac{\phi_3 - \phi_1}{\pi_3 - \pi_1} \qquad \theta_B = -\frac{1}{c_p}\,\frac{\phi_4 - \phi_2}{\pi_4 - \pi_1}$$

π stands for the Exner function, $\left(\frac{p}{P}\right)^{R/c_p}$.

$$\pi_A \equiv \left(\frac{p_A}{P}\right)^{R/c_p}$$

The choice of p_A is arbitrary but it must satisfy the conditions

$$p_1 < p_A < p_3$$

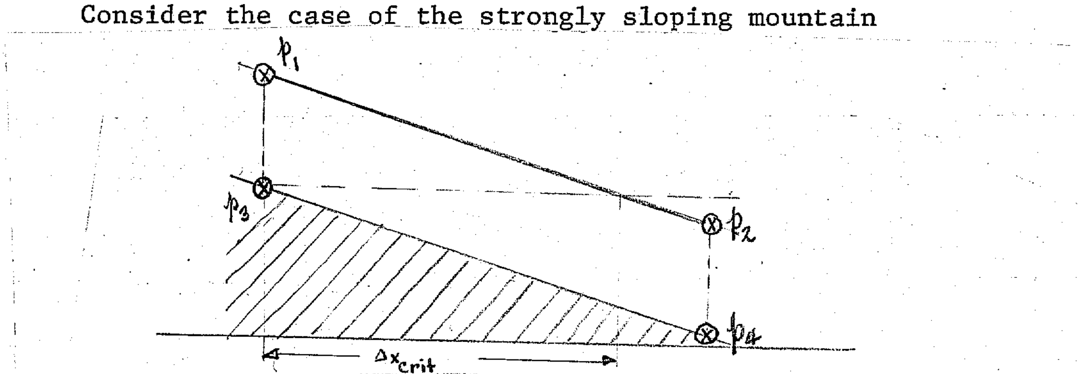
$$p_2 < p_A < p_4$$

if the above figure is to be a correct representation. Because of the hydrostatic assumption, one is assured that

$$p_1 < p_3 \quad \text{and} \quad p_2 < p_4$$

There is however no a priori guarantee that $p_2 < p_3$, or conversely that $p_1 < p_4$.

Consider the case of the strongly sloping mountain



If sloping terrain is to be used, it is necessary that the grid points not be separated by more than Δx_{crit}, if one is to use interpolational approximation for the pressure gradient force.

It should be noted that by taking "thicker" σ-layers, one may relax Δx_{crit}.

A conclusion to be drawn is that the horizontal mesh must be reduced in size if more σ layers are introduced into a model. A second conclusion is that for a fixed horizontal and vertical mesh there is a limit on the amount of slope permitted in the representation of orography (this also applies to the "tropopause" surface in the NMC models).

3.0 A Constrained Interpolation Scheme Compared with the NMC Scheme

Let us consider a specific interpolational formula for the pressure gradient force. Allow

$$p_A = \sigma_{i+\frac{1}{2}} \overline{p_\sigma}^x + \overline{p^*}^x = \overline{p}^{x\sigma}$$

where a Shuman type σ coordinate is assumed and $\sigma_{i+\frac{1}{2}} = \frac{1}{2}(\sigma_i + \sigma_{i+1})$. p^* is the pressure at $\sigma = 0$. Note that the constraints for p_A lying in the intervals

$$p_1 < p_A < p_3$$

$$p_2 < p_A < p_4$$

are more severe than those quoted earlier, but are of essentially the same type.

Use of the hydrostatic relationship

$$\frac{\Delta\phi}{\Delta\pi} = - c_p\theta$$

gives

$$\phi_B = \phi_2 - c_p \theta_B(\pi_A - \pi_2)$$

$$\phi_C = \phi_1 - c_p \theta_C(\pi_A - \pi_1)$$

$$\phi_B = \phi_4 + c_p \theta_B(\pi_4 - \pi_A)$$

$$\phi_C = \phi_3 + c_p \theta_C(\pi_3 - \pi_A)$$

The pressure gradient force on the $p = p_A$ surface is

$$F = \frac{\phi_B - \phi_C}{\Delta x}$$

Using the relations above, one may form two approximations for F

$$F_1 = \frac{\phi_2 - \phi_1}{\Delta x} - c_p\theta_B \frac{(\pi_A - \pi_2)}{\Delta x} + c_p\theta_C \frac{(\pi_A - \pi_1)}{\Delta x}$$

$$F_2 = \frac{\phi_4 - \phi_3}{\Delta x} + c_p\theta_B \frac{\pi_4 - \pi_A}{\Delta x} - c_p\theta_C \frac{(\pi_3 - \pi_A)}{\Delta x}$$

If these are averaged

$$F = \tfrac{1}{2}(F_1 + F_2) = \overline{\phi}^{\sigma}_x + \frac{c_p\theta_B}{2}\left[\frac{\pi_4 - \pi_A - \pi_A + \pi_2}{\Delta x}\right]$$

$$+ \frac{c_p\theta_C}{2}\left[\frac{\pi_A - \pi_1 - \pi_3 + \pi_A}{\Delta x}\right]$$

$$F = \overline{\phi}^{\sigma}_x + c_p\pi_A \frac{\theta_C - \theta_B}{\Delta x} - c_p \frac{\theta_C\overline{\pi}^{\sigma}_C - \theta_B\overline{\pi}^{\sigma}_B}{\Delta x}$$

or

$$F = \overline{\phi}^{\sigma}_x + c_p\overline{\theta}^{x}\,\overline{\pi}^{\sigma}_x + c_p[\overline{\pi}^{\sigma x} - \pi_A]\theta_x$$

It will be noted that the approximation for F thus derived differs from the usual NMC approximation in the appearance of the last term,

$$c_p\left(\overline{\pi}^{\sigma x} - \pi_A\right)\theta_x .$$

This term vanishes if θ_x does, but such is not generally the case. We can make the term vanish by judicious selection of p_A! This will generally be a choice other than that made initially but, except for the satisfaction of constraints related to the interpolational character of the approximation, the particular choice of p_A was not relevant to the derivation.

One requires

$$\pi_A = \overline{\pi}^{\sigma x}$$

or

$$\left(\frac{p_A}{P}\right)^{\kappa} = \tfrac{1}{4}\left(\left(\frac{p_2}{P}\right)^{\kappa} + \left(\frac{p_4}{P}\right)^{\kappa} + \left(\frac{p_1}{P}\right)^{\kappa} + \left(\frac{p_3}{P}\right)^{\kappa}\right)$$

We conclude that the NMC approximation

$$F = \overline{\phi}_x^{\sigma} + c_p\, \overline{\theta}^{x}\, \overline{\pi}_x^{\sigma}$$

is a consistent interpolational approximation if, and only if

$$p_A' \equiv \left(\tfrac{1}{4}\left(p_1^{\kappa} + p_2^{\kappa} + p_3^{\kappa} + p_4^{\kappa}\right)\right)^{\frac{1}{\kappa}}$$

satisfies the constraints

$$p_1 < p_A' < p_3$$

$$p_2 < p_A' < p_4$$

If this <u>is not</u> satisfied, the alternative

$$p_A = \overline{p^* + \sigma_{i+\frac{1}{2}}\, p_\sigma}^{x} = \tfrac{1}{4}\left(p_1 + p_2 + p_3 + p_4\right) = \overline{p}^{x\sigma}$$

may be tested; if this <u>is</u> satisfied, then the correction term

$$c_p\left(\overline{\pi}^{\sigma x} - \left(\frac{\overline{p}^{x\sigma}}{P}\right)^{\kappa}\right)\theta_x$$

should be introduced into the pressure gradient force approximation.

It is possible of course that neither set of constraints are satisfied, in which case neither approximation constitutes an interpolation.

4.0 An Unconditionally Interpolational Approximation

There is another technique for approximation of the pressure gradient force in the σ-layer which is not subject to the slope constraints discussed above.

One may use the differential formulas for the pressure gradient force on a σ-surface and directly approximate it by finite differences. The potential temperature coefficient required in the formula can be interpolated from the layer above and that below the σ surface except at the top and bottom of the model atmosphere. The problem is alleviated at the top of NMC models by the autobarotropic nature assumed for the layer. At the bottom of the model atmosphere, one may have recourse to a number of schemes for estimation of a surface potential temperature.

The formulation of the pressure gradient force in the layer would be:

$$F = \overline{\phi}^{\sigma}_{x} + c_p \, \overline{\overline{\theta}^{\sigma x} \, \pi_x}^{\sigma}$$

One may show that the difference between this approximation and the present NMC approximation is

$$F_{NEW} - F_{NMC} = c_p \overline{\left(\overline{\theta}^{\sigma\sigma} - \theta\right)}^{x} \, \overline{\pi}^{\sigma}_{x} + \left(\frac{\Delta\sigma}{2}\right)^2 c_p \, \overline{\theta}^{\sigma x}_{\sigma} \, \pi_{\sigma x}$$

$$= c_p \left(\frac{\Delta\sigma}{2}\right)^2 \left(\overline{\theta}^{x}_{\sigma\sigma} \, \overline{\pi}^{\sigma}_{x} + \overline{\theta}^{\sigma x}_{\sigma} \, \pi_{\sigma x}\right)$$

$$= c_p \left(\frac{\Delta\sigma}{2}\right)^2 \left(\overline{\theta}^{x}_{\sigma} \, \pi_x\right)_{\sigma}$$

Hence, the two approximations possess a potentially important difference related to the vertical variation of the static stability and pressure gradient along the σ surfaces.

The principal advantage of the new approximation seems to lie in its generality and independence of constraints with respect to interpolational legitimacy.