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COMPARATIVE ANALYSIS OF A NEW INTEGRATION METHOD WITH CERTAIN STANDARD METHODS

by

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Recently, experiments have been made with a new numerical scheme for integrating the primitive equations. The new method may be expressed by reference to the wave equation

\[ \frac{2\zeta}{\partial t} = i\omega \zeta \]  

by writing

\[ \zeta^{n+1} = \zeta^{n-1} + 2\Delta t i \omega \zeta^n \]  

\[ \zeta^n = \alpha \zeta^n + 0.5(1-\alpha)(\zeta^{n-1} + \zeta^{n+1}) \]  

in which the index, \( n \), fixes the time level and \( \alpha \) is a fraction less than unity.

When \( \alpha \) is set to unity, the scheme is the well-known "leapfrog" method. When \( \alpha \) is set to zero, the method reduces to one studied by Kurihara [1] and called by him the "leapfrog-backward" method. To show this last point, (2b) may be rewritten as \( \alpha = 0 \)

\[ \zeta^{n+1} = 0.5(\zeta^n + \zeta^n + 2\Delta t i \omega \zeta^{n+1}) \]

or

\[ \zeta^{n+1} = \zeta^n + \Delta t i \omega \zeta^{n+1} \]  

If one defines \( b = \omega\Delta t \), following Kurihara, the stability criterion for the leapfrog scheme is

\[ b \leq 1 \]

and for the leapfrog-backward scheme is

\[ b \leq 0.8 \]

Two other schemes have been used in numerical integrations of the primitive equations and analyzed by Kurihara. These are the Euler-backward scheme

\[ \zeta^{n+1} = \zeta^n + \Delta t i \omega \zeta^n \]  

\[ \zeta^{n+1} = \zeta^n + \Delta t i \omega \zeta^{n+1} \]
for which the stability criterion is

\[ b < 1. \]

and the "leapfrog-trapezoidal" method

\[ \zeta_{n+1}^* = \zeta_{n-1} + 2\Delta t \ i \ \omega \ \zeta^n \]  
\[ \zeta_{n+1} = \zeta^n + .5 \Delta t(i \ \omega \ \zeta^n + i \ \omega \ \zeta_{n+1}^*) \]  

One may show that the general scheme (2) provides a solution, \( \zeta_n^\alpha \), of the form

\[ \zeta_n^\alpha = (1-\alpha)\zeta_{L.B.}^n + \alpha \zeta_L^n \]  

where \( \zeta_{L.B.}^n \) is the result of integration with the leapfrog-backward method, and \( \zeta_L^n \) is the result of integration with the leapfrog method.

Now, interest has been expressed in the results to be expected with the method (2) for a variety of values of \( \alpha \). It should be noted that (5) does not necessarily imply stability of the new method whenever the criteria for the leapfrog-backward and leapfrog methods are satisfied separately. Therefore, we made calculations to solve the initial value problem,

\[ \frac{\partial \zeta}{\partial t} = i \ \omega \ \zeta \]  
\[ \zeta \text{ at } t = 0 \text{ is } \hat{\zeta} = 1 + 0 i, \]

with each of the methods discussed above. The starting procedure for use with method (2) was

\[ \zeta_1^* = \hat{\zeta} + i \ \omega \ \Delta t \ \hat{\zeta} \]  
\[ \zeta_0 = \alpha \hat{\zeta} + .5(1-\alpha)(\hat{\zeta} + \zeta_1^*) \]

We defined

\[ R = \frac{2\pi}{\omega \Delta t} \]
which implies that the period of the wave is \( R \) intervals of time measured in \( \Delta t \)-units. The amplitude of the solution after 15 steps is tabulated below for various values of \( R \) and \( \alpha \):

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>50</th>
<th>100</th>
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<td>&gt;100</td>
<td>1.22</td>
<td>1.16</td>
<td>1.03</td>
<td>1.07</td>
<td>1.08</td>
<td>1.02</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>.999</td>
<td>&gt;100</td>
<td>1.21</td>
<td>1.15</td>
<td>1.03</td>
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<td>1.08</td>
<td>1.02</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
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<tr>
<td>.990</td>
<td>&gt;100</td>
<td>1.15</td>
<td>1.12</td>
<td>1.03</td>
<td>1.06</td>
<td>1.07</td>
<td>1.02</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>.900</td>
<td>&gt;100</td>
<td>.97</td>
<td>.94</td>
<td>.99</td>
<td>.97</td>
<td>1.00</td>
<td>1.00</td>
<td>.99</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
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<td>&gt;100</td>
<td>.56</td>
<td>.71</td>
<td>.79</td>
<td>.84</td>
<td>.88</td>
<td>.90</td>
<td>.92</td>
<td>.99</td>
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<tr>
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<td>.32</td>
<td>.49</td>
<td>.61</td>
<td>.69</td>
<td>.75</td>
<td>.79</td>
<td>.97</td>
<td>.99</td>
</tr>
<tr>
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<td>&gt;100</td>
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<td>.11</td>
<td>.21</td>
<td>.36</td>
<td>.48</td>
<td>.57</td>
<td>.64</td>
<td>.94</td>
<td>.99</td>
</tr>
<tr>
<td>0.0</td>
<td>&gt;100</td>
<td>&gt;100</td>
<td>4.42</td>
<td>.11</td>
<td>.12</td>
<td>.25</td>
<td>.36</td>
<td>.45</td>
<td>.89</td>
<td>.97</td>
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<tr>
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<td>&gt;100</td>
<td>&gt;100</td>
<td>35.44</td>
<td>2.74</td>
<td>.34</td>
<td>.06</td>
<td>.12</td>
<td>.22</td>
<td>.83</td>
<td>.96</td>
</tr>
</tbody>
</table>

| E.B.      | 2.13 | .13  | .13  | .19  | .27  | .35  | .43  | .50  | .89  | .97  |
| L.T.      | .23  | .56  | .75  | .86  | .91  | .94  | .96  | .97  | 1.00 | 1.00 |

It will be noted that the leapfrog method yields amplitudes greater than unity even for \( R > 2\pi \), the computational stability criterion corresponding to \( b \leq 1 \). This error is associated with the amplification produced by the "forward," starting scheme. It will be noted that that error is greatly reduced by using \( \alpha = .90 \). The empirical result for \( \alpha = 0 \), suggests that the instability with \( R = 8,10 \) (should be stable by Kurihara's result when \( b < .8, R = 8 \)) is also related to the "forward" start utilized with that method (see eqs. 8) and the greater weight attached to the amplified value of \( \zeta^1 \).

Since both the leapfrog-trapezoidal and Euler-backward methods require the computation of two tendencies to advance the calculation, the scheme with \( \alpha = .9 \) or .75 seems to have considerable merit from an efficiency viewpoint.