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Potential Initialization: An Implicit Version of Normal Mode Initialization

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1. Introduction

This note describes a procedure exactly equivalent to normal mode initialization which does not require explicit mention of normal modes. Mode space equations are replaced by simple differential equations which relate geostrophic and ageostrophic components of the model initial state.

The method is based on representation of geopotential and horizontal wind vector by three scalar fields: (1) the geostrophic potential, from which the geostrophic part of the wind and geopotential fields are obtained; (2) the rotational ageostrophic potential, which defines the rotational part of the ageostrophic wind, and also contributes to the geopotential and (3) the divergent ageostrophic potential, which takes care of the divergent wind but has no geopotential component. The ageostrophic component, so defined, does not interact with the geostrophic component, and can therefore be used as the basis for non-linear initialization in a manner similar to that used with normal modes. Fourier analysis shows that the procedures are identical. The non-linear procedure developed here uses the small-parameter expansion of Baer (1977), with a modification that makes the procedure less cumbersome to apply. In terms of the manifold concept introduced by Leith (1980), the geostrophic potential defines the "slow mode axis", and the ageostrophic potentials the "fast mode axis". The ageostrophic potentials computed from a given geostrophic potential via the small parameter expansion define a point on the slow manifold.

In addition to giving a clearer understanding of normal mode initialization, this formulation can be applied readily to limited area models. There is a significant difference between this procedure when applied to a limited area model, and those suggested by Machenhauer (personal communication) and Briere (1981). In order to define reasonable normal mode expansion functions, unnecessarily restrictive boundary conditions must be specified. With the differential equations derived here, no such functions are necessary. More natural boundary conditions can be specified for solution of these equations. To see how this is done, the potential representation is applied to a limited area barotropic shallow water model with orography. Computational results will be presented in a later note.

2. The Potential Field Representation

The linear constant f plane shallow water equations can be written

$$u_t = v - \phi_x \tag{2.1}$$

$$v_t = -u - \phi_y \tag{2.2}$$

$$\phi_t = -u_x - v_y \tag{2.3}$$

where t is scaled by f_o^{-1}, (x, y) by cf_o^{-1} and ϕ by c. $f_o = 2\Omega_o$ is twice the rotation rate, and $c = \sqrt{gh_o}$ is the linear gravity wave phase speed (h_o the mean depth). Then u, v, and ϕ have units of speed.

We suppose that u, v, ϕ can be related to three scalar fields, S, W, P as follows:

$$u = -S_y - W_y + P_x \equiv u_s + u_w + u_p \tag{2.4}$$

$$v = S_x + W_x + P_y \equiv v_s + v_w + v_p \tag{2.5}$$

$$\phi = L_s S + L_w W + L_p P \equiv \phi_s + \phi_w + \phi_p \tag{2.6}$$

where L_s, L_w, L_p are as yet unspecified linear differential operators. S represents the geostrophic component of the flow, while W and P are designated for the rotational and divergent components of the ageostrophic flow. Then clearly $L_s = 1$ yields the proper relation for u_s, v_s, ϕ_s, viz

$$u_s = -S_y \tag{2.7}$$

$$v_s = S_x \tag{2.8}$$

$$\phi_s = S \tag{2.9}$$

To determine L_w and L_p, we require that $\frac{\partial S}{\partial t} \propto S$. Then, if (2.1)-(2.3) are initialized with a geostrophic state u_s, v_s, ϕ_s, no ageostrophic component will be generated during time integration.

The strategy then is to first substitute (2.4)-(2.6) into (2.1)-(2.3) and then solve for S_t, W_t and P_t. In terms of S, W, P, (2.1)-(2.3) become

$$-S_{yt} - W_{yt} + P_{xt} = W_x + P_y - L_w W_x - L_p P_x \tag{2.10}$$

$$S_{xt} + W_{xt} + P_{yt} = W_y - P_x - L_w W_y - L_p P_y \tag{2.11}$$

$$S_t + L_w W_t + L_p P_t = -\nabla^2 P \tag{2.12}$$

It is assumed in advance that L_w, L_p, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are all interchangeable operations. This must be verified after L_w and L_p have been specified. Taking $\frac{\partial}{\partial x}$ of (2.10) added to $\frac{\partial}{\partial y}$ of (2.11) gives the divergence equation

$$\nabla^2 P_t = \nabla^2 (1 - L_w) W - L_p \nabla^2 P \tag{2.13}$$

Similiarily, $\frac{\partial}{\partial x}$ of (2.11) minus $\frac{\partial}{\partial y}$ of (2.10) gives the vorticity equation

$$\nabla^2 S_t + \nabla^2 W_t = -\nabla^2 P \tag{2.14}$$

We now have solved for P_t in (2.13). If we choose $L_p = 0$, then equations (2.12) and (2.14) can be solved for S_t and W_t. There results

$$\nabla^2(1-L_w)W_t = -(\nabla^2-\nabla^4)P \tag{2.15}$$

$$\nabla^2(1-L_w)S_t = -\nabla^2(\nabla^2-L_w)P \tag{2.16}$$

To eliminate dependence of S_t on W or P, we must choose $L_w = \nabla^2$. So finally we have the desired representation:

$$u = -S_y - W_y + P_x \tag{2.17}$$

$$v = S_x + W_x + P_y \tag{2.18}$$

$$\phi = S + \nabla^2 W \tag{2.19}$$

and the tendency equations for S, W, P become

$$(\nabla^4-\nabla^2)S_t = 0 \tag{2.20}$$

$$(\nabla^4-\nabla^2)W_t = -(\nabla^4-\nabla^2)P \tag{2.21}$$

$$\nabla^2 P_t = -(\nabla^4-\nabla^2)W \tag{2.22}$$

To obtain S, W, P give u, v, ϕ we solve the system of equations

$$\nabla^2\psi = v_x - u_y \tag{2.23}$$

$$\nabla^2 P = u_x + v_y \tag{2.24}$$

$$\nabla^2 W - W = \phi - \psi \tag{2.25}$$

$$S = \psi - W \tag{2.26}$$

In solving (2.23)-(2.26) there is a problem deciding what to use for boundary conditions when u, v, ϕ are defined on a limited domain. This is similar to the problem of solving the Helmholtz equations for rotational and divergent potentials. A procedure routinely used at NMC (subroutine HANS, see Gerrity (1976) for details) has been adapted to solve (2.23)-(2.24) for Ψ and P. A simple modification to HANS makes it suitable for obtaining W from (2.25).

3. Small Parameter Expansion

We now apply the potential representation to initialization. To this end, the small parameter expansion of Baer (1977) is used instead of Machenhauer's (1977) iterative scheme. There are several reasons for this. Phillips (1981) has pointed out that convergence of the Machenhauer procedure is a function of, among other things, the amplitude of the mean flow component, while the small parameter expansion converges independent of the mean flow. Even when Machenhauer converges, Phillips demonstrated that it can converge to an incorrect result. Much difficulty has been experienced in global models initialized with normal mode initialization using the Machenhauer iteration. Physics cannot be included, initialization can only be applied to the first few largest vertical wavelengths, and no improvement in forecast skill has been demonstrated. So there seems to be little point in pursuing this approach for a limited area model, where the short range forecast is to be improved. Accordingly we look at the small parameter expansion, in the hope that real improvement in forecast skill for the limited area type of model can eventually be achieved through more accurate specification of initial conditions.

Suppose $\varepsilon \ll 1$ is a dimensionless parameter and we scale u, v, ϕ by ε, and t by ε^{-1}, then the non-linear f-plane equations may be written

$$\varepsilon u_t = v - \phi_x + \varepsilon u^a \tag{3.1}$$

$$\varepsilon v_t = -u - \phi_y + \varepsilon v^a \tag{3.2}$$

$$\varepsilon \phi_t = -u_x - v_y + \varepsilon \phi^a \tag{3.3}$$

where u^a, v^a, ϕ^a are used to represent non-linear forcing. In the next section, u^a, v^a, ϕ^a are given specific forms for a barotropic model with orography, that will serve as a computational example.

In terms of the S, W, P representation just introduced (3.1)-(3.3) become:

$$S_t = S^a \tag{3.4}$$

$$\varepsilon W_t = -P + \varepsilon W^a \tag{3.5}$$

$$\varepsilon P_t = -(\nabla^2 - 1) W + \varepsilon P^a \tag{3.6}$$

Note that (3.4)-(3.6) are integrated versions of (2.20)-(2.22) with non-linear terms added. $(\nabla^4 - \nabla^2)$ has been removed from (3.4)-(3.5), and ∇^2 from (3.6). This could be cause for concern. However, any solution which satisfies (3.4)-(3.6) will also satisfy the differentiated equations.

Now the initialization problem is--given S, determine W and P such that the resulting time evoluation is "slow", i.e. $(W_t, P_t) = 0(\varepsilon)$. Define

$$(S, W, P) = \sum_{n=0}^{\infty} \varepsilon^n (S_n, W_n, P_n) \tag{3.7}$$

Substituting (3.7) into (3.4)-(3.6) and equating equal powers of ε yields for the nth order system

$$S_{nt} = S_n^a \tag{3.8}$$

$$W_{n-1,t} = -P_n + W_{n-1}^a \tag{3.9}$$

$$P_{n-1,t} = -(\nabla^2 - 1) W_n + P_{n-1}^a \tag{3.10}$$

The zero order solution is

$$S_{ot} = S_o^a \tag{3.11}$$

$$P_o = 0 \tag{3.12}$$

$$W_o = 0 \tag{3.13}$$

So S_o is arbitrary, but $P_o = W_o = 0$. Now we assume $S_n = 0$ for $n > 0$, and then we have for P_n, W_n

$$P_n = W_{n-1}^a - W_{n-1,t} \tag{3.14}$$

$$(\nabla^2 - 1) W_n = P_{n-1}^a - P_{n-1,t} \tag{3.15}$$

$$S_{nt} = S_n^a \tag{3.16}$$

To determine P_n, W_n, we must compute, from $(P_j, W_j, S_{j,t}; j=1, n-1)$ and $(P_{j,t}, W_{j,t}; j=1, n-2)$ W_{n-1}^a, $W_{n-1,t}$ and P_{n-1}^a, $P_{n-1,t}$. This is rather difficult to do when W^a, P^a, S^a are non-linear functions of S, W, P. To first order, it is not very difficult (equivalent in fact to one interation of Machenhauer), while second order is more of a problem and higher order virtually impossible. However, the effect of terrain, latent heating, surface friction, and model geometry can require several orders of solution before adequate convergence is achieved. This computational difficulty is perhaps the principle reason why the expansion method is not yet implemented in practice.

To solve this dilemma, linearize S_n^a, W_n^a, P_n^a about the zero order state $(S_o, 0, 0)$:

$$\left.\begin{aligned} S_n^a &= S_n^L + S_n^{NL} \\ W_n^a &= W_n^L + W_n^{NL} \\ P_n^a &= P_n^L + P_n^{NL} \end{aligned}\right\} \tag{3.17}$$

where S_n^L, W_n^L, P_n^L now depend linearly on S_n, W_n, P_n and S_n^{NL}, W_n^{NL}, P_n^{NL} non-linearly on S_j, W_j, P_j for $1 \leq j \leq n$. By definition $S_o^{NL} = W_o^{NL} = P_o^{NL} = 0$. Then

$$\left.\begin{aligned} S_{o*} &= S_o^L \\ P_o &= 0 \\ W_o &= 0 \\ S_{n*} &= S_n^L + S_n^{NL} \\ (\nabla^2 - 1) W_n &= P_{n-1}^L - P_{n-1,*} + P_{n-1}^{NL} \\ P_n &= W_{n-1}^L - W_{n-1,*} + W_{n-1}^{NL} \end{aligned}\right\} \tag{3.18}$$

To first order, the non-linear part of the solution is not involved. We have

$$(\nabla^2 - 1) W_1 = P_o^L \tag{3.19}$$

$$P_1 = W_o^L \tag{3.20}$$

The second order solution contains the first contribution from non-linear terms:

$$(\nabla^2 - 1)\, W_2 = P_1^{L} - P_{1t} + P_1^{NL} \tag{3.21}$$

$$P_2 = W_1^{L} - W_{1t} + W_1^{NL} \tag{3.22}$$

Define

$$(\nabla^2 - 1)\, W^* = P_1^{NL} \tag{3.23}$$

$$P^* = W_1^{NL} \tag{3.24}$$

To compute W_1^{NL}, P_1^{NL}, we first obtain u_1^{NL}, v_1^{NL}, ϕ_1^{NL} using u_1, v_1, ϕ_1 (obtained from W_1, P_1). Then obtain S_1^{NL}, W_1^{NL}, P_1^{NL} by solving (2.23)-(2.26). Now compute the linear part of the solution (neglecting the NL terms) until $\| W_n, P_n \| < \alpha \| W^*, P^* \|$.
The assumption is made that $\| W_n^{NL}, P_n^{NL} \| \ll \| W_n^{L}, P_n^{L} \|$ and that the cumulative effect of neglecting these terms will not be important until a fairly high order n is reached in the solution process.

It is much easier to calculate W_n^L, P_n^L than W_n^a, P_n^a. The following recursion is useful in computing the required terms. Using the notation $A_n^{\ell} = \dfrac{\partial^{\ell} A_n}{\partial t^{\ell}}$ and $0 = (\nabla^2 - 1)$, then we have

$$S_k^{\ell-k+1} = S^L\left(S_k^{\ell-k}, W_k^{\ell-k}, P_k^{\ell-k}\right) \tag{3.25}$$

$$W_{k+1}^{\ell-k} = 0^{-1}\left\{P^L\left(S_k^{\ell-k}, W_k^{\ell-k}, P_k^{\ell-k}\right) - P_k^{\ell+1-k}\right\} \tag{3.26}$$

$$P_{k+1}^{\ell-k} = W^L\left(S_k^{\ell-k}, W_k^{\ell-k}, P_k^{\ell-k}\right) - W_k^{\ell+1-k} \tag{3.27}$$

We start the recursion at $\ell = 0, k = 0$ with S_0^o the given zero-order state, and $P_0^o = W_0^o = 0$. Then for each value ℓ, $0 \le \ell \le N$, we evaluate (3.25)-(3.27) for $0 \le k \le \ell$. The result at stage ℓ is

$$S_1^{\ell} = \frac{\partial S_\ell}{\partial t}$$

$$P^{o}_{\ell+1} = P_{\ell+1}$$

$$W^{o}_{\ell+1} = W_{\ell+1}$$

which gives the $(\ell+1)$ terms of W and P.

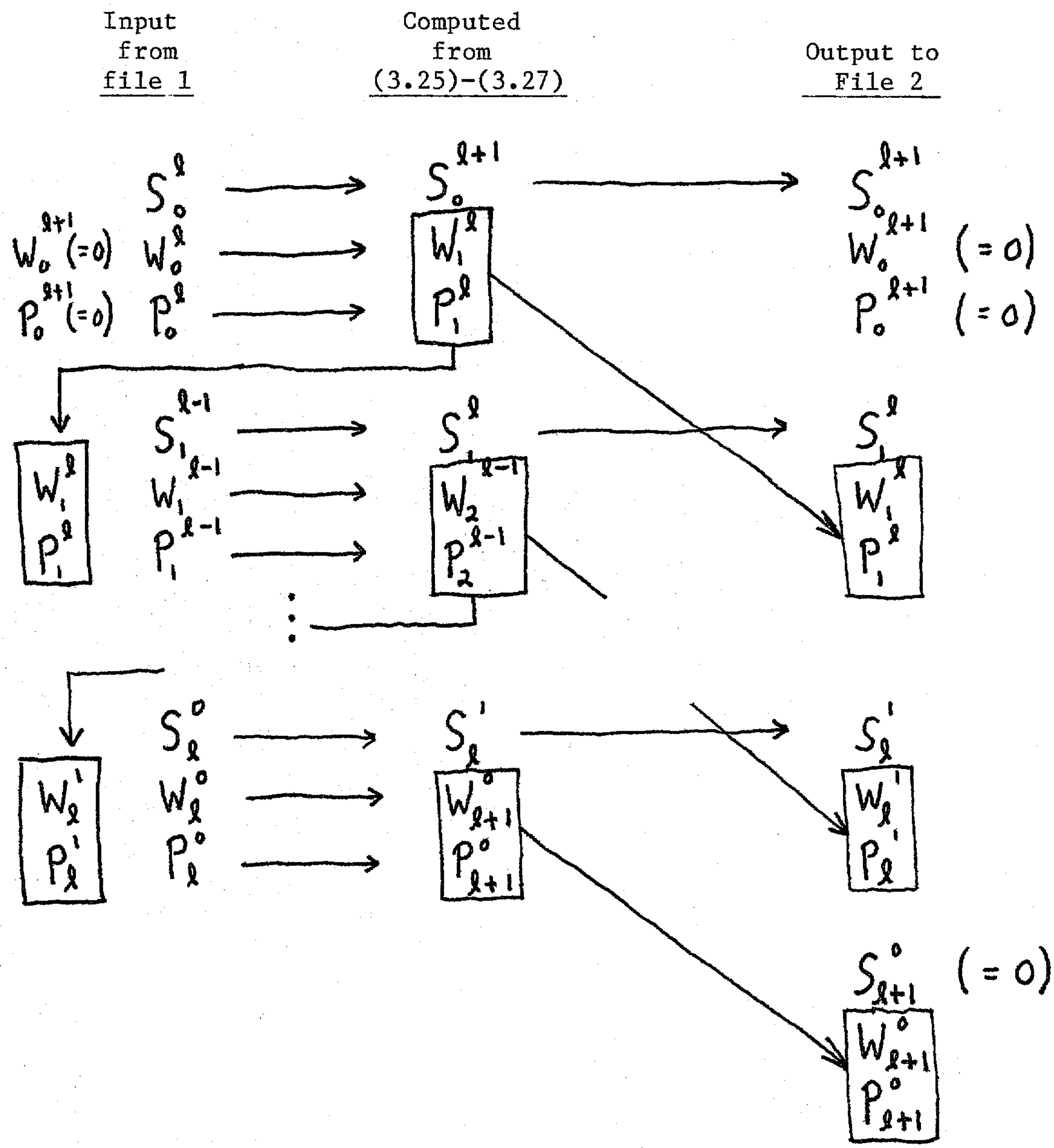
Some explanation of (3.25)-(3.27) is in order. The terms

$$S^L(S,W,P)\;;\; W^L(S,W,P)\;;\; P^L(S,W,P)$$

are obtained by first computing u, v, ϕ from S, W, P using the definition (2.17)-(2.19), then computing u^L, v^L, ϕ^L, the parts of u^a , v^a, ϕ^a linearized about u_o, v_o, ϕ_o. Finally we solve for S^L, W^L, P^L using (2.23)-(2.26). The inversion of $0 = (\nabla^2 - 1)$ represented in (3.26) is identical to solving (2.25) (the modified HANS subroutine is used). The only disadvantage of the recursion scheme presented here is that a large number of intermediate fields must be saved. A storage scheme has been worked out that would require the use of two disk files. Figure 2.1 illustrates the scheme, and also gives a better picture of the pattern generated by (3.25)-(3.27).

Now we consider a specific example.

Stage ℓ:



Initialize file 1 for $\ell = 0$ with S_0^0, W_0^0, P_0^0 when stage ℓ is complete, $\ell = \ell + 1$, and file 2 becomes input file, file 1 the output file.

Figure 2.1

4. Barotropic model

As a specific example of application of the previously outlined initialization procedure, we consider a barotropic model on a polar stereographic projection with the effect of orography included. The equations are:

$$\frac{\partial}{\partial t}\left(\frac{u}{m}\right) = \eta \frac{v}{m} - \frac{\partial \phi}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}\left(u^2 + v^2\right) \tag{4.1}$$

$$\frac{\partial}{\partial t}\left(\frac{v}{m}\right) = -\frac{\eta u}{m} - \frac{\partial \phi}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}\left(u^2 + v^2\right) \tag{4.2}$$

$$\frac{\partial \phi}{\partial t} = -m^2\left\{\frac{\partial}{\partial x}\left[\frac{u}{m}\left(\phi + \bar{\Phi} - \phi_0\right)\right] + \frac{\partial}{\partial y}\left[\frac{v}{m}\left(\phi + \bar{\Phi} - \phi_0\right)\right]\right\} \tag{4.3}$$

$$\eta = f + \zeta \tag{4.4}$$

$$\zeta = m^2\left[\frac{\partial}{\partial x}\left(\frac{v}{m}\right) - \frac{\partial}{\partial y}\left(\frac{u}{m}\right)\right] \tag{4.5}$$

$$m = \frac{1 + \sin\left(\frac{\pi}{3}\right)}{1 + \sin \phi_e} \tag{4.6}$$

$$f = 2\Omega\left(\frac{1 - R}{1 + R}\right) \tag{4.7}$$

$$R = \frac{x^2 + y^2}{\left[a\left(1 + \sin\frac{\pi}{3}\right)\right]^2} \tag{4.8}$$

Ω = 7.292×10^{-5} sec^{-1}

a = 6.371×10^{6} m

ϕ = deviation geopotential

$\bar{\Phi}$ = $g\bar{h}$ - rest state geopotential

ϕ_0 = gh_o ground geopotential

To get these equations into the form of (3.1)-(3.3), we do the following. First define scaled winds

$$(u^*, v^*) = \frac{f}{m f_0}(u, v) \qquad (f_0 = 2\Omega) \tag{4.9}$$

Then (4.1)-(4.3) may be written

$$\frac{\partial u^*}{\partial t} = f_0 v^* - \frac{\partial \phi}{\partial x} + f_0 u^a \tag{4.10}$$

$$\frac{\partial v^*}{\partial t} = -f_0 u^* - \frac{\partial \phi}{\partial y} + f_0 v^a \tag{4.11}$$

$$\frac{\partial \phi}{\partial t} = -c^2\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) + f_0 \phi^a \tag{4.12}$$

where $C^2 = \bar{\Phi}$ and

$$f_0 u^a = \left(\frac{f}{f_0} - 1\right)\frac{\partial}{\partial t}\left(\frac{u}{m}\right) + \frac{\zeta v}{m} - \frac{1}{2}\frac{\partial}{\partial x}\left(u^2 + v^2\right) \tag{4.13}$$

$$f_0 v^a = \left(\frac{f}{f_0} - 1\right)\frac{\partial}{\partial t}\left(\frac{v}{m}\right) - \frac{\zeta u}{m} - \frac{1}{2}\frac{\partial}{\partial y}\left(u^2 + v^2\right) \tag{4.14}$$

$$f_0 \phi^a = \frac{\bar{\Phi}}{f_0}\left[\frac{\partial}{\partial x}\left(\frac{fu}{m}\right) + \frac{\partial}{\partial y}\left(\frac{fv}{m}\right)\right] - m^2\left\{\frac{\partial}{\partial x}\left[\frac{u}{m}\left(\phi + \bar{\Phi} - \phi_0\right)\right] + \frac{\partial}{\partial y}\left[\frac{v}{m}\left(\phi + \bar{\Phi} - \phi_0\right)\right]\right\} \tag{4.15}$$

Now apply the same scaling as used in section 2. The desired form is now

$$u_t = v - \phi_x + u^a \tag{4.16}$$

$$v_t = -u - \phi_y + v^a \tag{4.17}$$

$$\phi_t = -u_x - v_y + \phi^a \tag{4.18}$$

When referring to (4.16)-(4.18) the variables are assumed to be scaled by (4.9) and the non-dimesionalization of section 2. When referring to computation of the non-linear terms, the variables are assumed to be the original unscaled form. To get (4.13)-(4.15) in a form more convenient for computing, use (4.1)-(4.2) to replace $\frac{\partial}{\partial t}(u/m)$, $\frac{\partial}{\partial t}(v/m)$ in (4.13), (4.14):

$$f_o u^a = \left(\frac{f+\zeta}{f_o} - 1\right)\frac{fv}{m} + \left(1 - \frac{f}{f_o}\right)\frac{\partial \phi}{\partial x} - \frac{1}{2}\frac{f}{f_o}\frac{\partial}{\partial x}(u^2+v^2) \tag{4.19}$$

$$f_o v^a = -\left(\frac{f+\zeta}{f_o} - 1\right)\frac{fu}{m} + \left(1 - \frac{f}{f_o}\right)\frac{\partial \phi}{\partial y} - \frac{1}{2}\frac{f}{f_o}\frac{\partial}{\partial y}(u^2+v^2) \tag{4.20}$$

To obtain computational forms for u^L, v^L, ϕ^L and u^{NL}, v^{NL}, ϕ^{NL} as required for the procedure outlined in the previous section, replace u, v, ϕ by $u_o + u^1$, $v_o + v^1$, $\phi_o + \phi^1$. Then u^L, v^L, ϕ^L is that part of u^a, v^a, ϕ^a which is linear in u^1, v^1, ϕ^1. u^L, v^L, ϕ^L can most easily be computed by first obtaining u^{NL}, v^{NL}, ϕ^{NL}, where

$$f_o u^{NL} = \frac{\zeta'}{f_o}\frac{fv'}{m} - \frac{1}{2}\frac{f}{f_o}\frac{\partial}{\partial x}\left[(u')^2 + (v')^2\right] \tag{4.21}$$

$$f_o v^{NL} = -\frac{\zeta'}{f_o}\frac{fu'}{m} - \frac{1}{2}\frac{f}{f_o}\frac{\partial}{\partial y}\left[(u')^2 + (v')^2\right] \tag{4.22}$$

$$f_o \phi^{NL} = -m^2\left\{\frac{\partial}{\partial x}\left(\frac{u'\phi'}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v'\phi'}{m}\right)\right\} \tag{4.23}$$

Then we have

$$f_o u^L = f_o u^a - f_o u^{NL} \tag{4.24}$$

$$f_o v^L = f_o v^a - f_o v^{NL} \tag{4.25}$$

$$f_o \phi^L = f_o \phi^a - f_o \phi^{NL} \tag{4.26}$$

Now, when applying the recursion (3.25)-(3.27) to obtain a balanced initial state, u_o, v_o, ϕ_o will always be defined by $\overset{o}{S}_o$, $\overset{o}{P}_o = 0$, $\overset{o}{W}_o = 0$ and

$$u' = u_n^{\ell} \equiv \frac{\partial^{\ell} u_n}{\partial t^{\ell}}$$

$$v' = v_n^{\ell} \equiv \frac{\partial^{\ell} v_n}{\partial t^{\ell}}$$

$$\phi' = \phi_n^{\ell} = \frac{\partial^{\ell} \phi_n}{\partial t^{\ell}}$$

when computing u^L, v^L, ϕ^L from which S^L, W^L, P^L are obtained.

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