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OFFICE NOTE 169

Elemental-Filter Design Considerations

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ELEMENTAL-FILTER DESIGN CONSIDERATIONS

1. Introduction

In Office Note 165, Shuman noted that the G-filter (a seven-point operator) used in the LFM postprocessor can be simulated by using three 3-point symmetric smoothing elements of the type he developed in his 1957 Monthly Weather Review paper on smoothing and filtering in NWP.

In this note, I will show how the relation given by Shuman for the G-filter relates to the design criteria for that filter, developed in Office Note 57. I then show how the progression from the five-point operator, called the H-filter by Shuman, to the G-filter can be both extended and modified. On the way, we encounter complex roots for a filter response function which leads to an application of Shuman's method for avoiding complex weights through the use of antisymmetric three-point elemental operators.

Finally, I'll sketch a procedure for the calculation of the appropriate three-point symmetric or antisymmetric elements that satisfy a large set of conditions on the filter response function.

2. Relationship Between Expanded and Elemental Filters

Let d_j be a set of real numbers forming a data set on an equispaced array of points with indices j. Let w_m denote a set of real numbers, and let y_j be a set of real numbers formed from d_j and w_m by the relationship (1),

$$y_j = w_o\, d_j + \sum_{m=1}^{M} w_m(d_{j+m} + d_{j-m}). \qquad (1)$$

Provided that

$$w_o + \sum_{m=1}^{M} 2w_m = 1 \tag{2}$$

the linear transformation (1) may be called a filter. The properties of the transformation are readily interpreted if we select d_j and y_j to be represented by Fourier series. In view of (2) and the symmetry of (1), one may treat a single element of the Fourier series representations of y_i and d_j,

$$\begin{aligned} d_j &= D \cos kjh \\ y_j &= Y \cos kjh, \end{aligned} \tag{3}$$

in which h is the spacing between the data points and k is the wave number.

One may define

$$R(\cos kh) \equiv w_o + \sum_{m=1}^{M} 2w_m \cos(mkh) \tag{4}$$

and obtain from (3) and (1) the relation

$$\frac{Y}{D} = R(\cos kh) \tag{5}$$

R is called the filter response function.

It is convenient to define

$$\zeta = \cos kh \tag{6}$$

and to use the recurrence relationship,

$$\cos mkh = [2 \cos(m-1)kh]\zeta - [\cos(m-2)kh] \tag{7}$$

to rewrite (4) as

$$R(\zeta) = \sum_{m=o}^{M} c_m \zeta^m \tag{8}$$

The coefficients c_m are real valued and may be calculated from the w's.

The variable ζ is continuous; for our purposes its significant range is $-1 \leq \zeta \leq +1$. The value, $\zeta = -1$, is taken on when the function d has wave length 2h; the value, $\zeta = 0$, is taken on when d has wave length 4h; the value $\zeta \to +1$ when the wave length becomes very large.

From (8), it is apparent that one may generally write,

$$R(\zeta) = \bar{c}[(\zeta-r_1)(\zeta-r_2)\ldots(\zeta-r_M)] \tag{9}$$

In view of (2), one has $R(+1) = 1$ and

$$\bar{c} = [(1-r_1)(1-r_2)\ldots(1-r_M)]^{-1} \tag{10}$$

In view of the factored form (9), one is led to consider representing the filter (1) by a series of elemental filters, each of which produces one of the factors in (1). Following Shuman's 1957 paper, one may write the response of a three-point symmetric elemental filter as

$$\rho(\zeta) = \nu(\zeta - (\frac{\nu-1}{\nu})) \tag{11}$$

where ν is twice the weight assigned to the end points of the elemental filter. Comparison of (11) and (9) shows that to each root r there corresponds a weight ν,

$$\nu = (1-r)^{-1} \tag{12}$$

One notes, however, that the roots r may be complex numbers; thus, the ν's may also be complex.

Generally, the complex roots occur in conjugate pairs because the coefficients c_m are real. To avoid dealing with complex numbers, Shuman suggests (Office Note 125) the following method:

Let the complex conjugate roots be r, r^* and the complex conjugate weights ν, ν^* be

$$\begin{aligned} \nu &= A + i\,B \\ \nu^* &= A - i\,B \end{aligned} \tag{13}$$

with A and B real numbers.

Define

$$D \equiv [(1-2A)^2 + 4B^2]^{\frac{1}{2}} \tag{14}$$

and

$$b = \tfrac{1}{2}(1-D) \tag{15}$$

$$a = \frac{1+D}{4} + [\frac{1-2A}{8} + \frac{D}{8}]^{\frac{1}{2}} \tag{16}$$

$$c = \frac{1+D}{4} - [\frac{1-2A}{8} + \frac{D}{8}]^{\frac{1}{2}} \tag{17}$$

Note that $a + b + c = 1$, and that a, b, and c are real valued.

With these values of a, b, and c, Shuman shows that the filters

$$\begin{aligned} \overline{F}_j &= a\, f_{j-1} + b\, f_j + c\, f_{j+1} \\ \overline{\overline{F}}_j &= c\, \overline{F}_{j-1} + b\, \overline{F}_j + a\, \overline{F}_{j+1} \end{aligned} \tag{18}$$

have the response function,

$$S(\zeta) = \frac{\overline{\overline{F}}_j}{f_j} = \frac{1}{(1-r)(1-r^*)}(\zeta - r)(\zeta - r^*) \tag{19}$$

3. Filter Design Criteria

Since the action of a filter transformation can be expressed by a response function in polynomial form, one may set design criteria to be satisfied by the response function and fix the coefficients c_m in equation (8). Once the coefficients are fixed, the polynomial's roots may be determined.

From the roots r one obtains the weights ν from equation (12). When complex roots occur, one may use the antisymmetric elemental filters. Through the use of 14 through 17, one determines the real valued weights of the antisymmetric elemental filters.

Let's now consider the filters H and G. The H filter satisfies the constraints,

$$\begin{aligned} R\ (+1) &= 1 \\ R'(+1) &= 0 \\ R\ (-1) &= 0 \end{aligned} \tag{20}$$

Its response function may be written

$$R_H(\zeta) = -\tfrac{1}{4}(\zeta+1)(\zeta-3) \tag{21}$$

and its weights are $\nu_1 = \tfrac{1}{2}$, $\nu_2 = -\tfrac{1}{2}$.

The G-filter was designed to add suppression to the high wave number response by appending the constraint

$$R'(-1) = 0 \tag{22}$$

to those (20) of the H filter's design. One obtains the filter response function

$$R_G(\zeta) = -\tfrac{1}{4}(\zeta+1)^2\ (\zeta-2) \tag{23}$$

and the weights $\nu_1 = \tfrac{1}{2}$, $\nu_2 = \tfrac{1}{2}$, $\nu_3 = -1$, as pointed out by Shuman.

Further suppression of the high wave numbers may be obtained by requiring additional derivatives of R to vanish at $\zeta = -1$. The first m-1 derivatives of R at $\zeta = -1$ will be set to zero, while the other constraints of the H filter are still satisfied, by a filter with the

response function

$$R(\zeta) = -\left(\frac{1}{2}\right)^m \left(\frac{m}{2}\right)(\zeta+1)^m \left(\zeta - \frac{m+2}{m}\right) \tag{24}$$

with the weights, $\nu_1 = \nu_2 = \ldots = \nu_m = \frac{1}{2}$, $\nu_{m+1} = -\frac{m}{2}$.

Going back to the H filter, one might wish to improve its fidelity for long waves. Suppose for example one desired $R''(+1) = 0$ in addition to the constraints (20). We use the polynomial

$$R(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 \tag{25}$$

and the constraints

$$\begin{aligned} R(+1) &= 1 = c_0 + c_1 + c_2 + c_3 \\ R'(+1) &= 0 = \quad\;\; c_1 + 2c_2 + 3c_3 \\ R''(+1) &= 0 = \qquad\quad\;\; 2c_2 + 6c_3 \\ R(-1) &= 0 = c_0 - c_1 + c_2 - c_3 \end{aligned} \tag{26}$$

Solving (26) one obtains

$$c_0 = \frac{7}{8}$$

$$c_1 = \frac{3}{8}$$

$$c_2 = -\frac{3}{8}$$

$$c_3 = \frac{1}{8}$$

So the polynomial becomes

$$R(\zeta) = \frac{1}{8}\left(\zeta^3 - 3\zeta^2 + 3\zeta + 7\right) \tag{27}$$

One notes that $\zeta = -1$ is a root of (27), so

$$R(\zeta) = \frac{1}{8}(\zeta+1)(\zeta^2 - 4\zeta + 7) \tag{28}$$

or after factoring the quadratic,

$$R(\zeta) = \frac{1}{8}(\zeta+1)(\zeta - 2 + i\sqrt{3})(\zeta - 2 - i\sqrt{3}). \tag{29}$$

Thus we encounter complex conjugate roots. The complex conjugate weights are

$$\begin{aligned} \nu &= -\left(\frac{1}{4} + i\frac{\sqrt{3}}{4}\right) \\ \nu^* &= -\left(\frac{1}{4} - i\frac{\sqrt{3}}{4}\right) \end{aligned} \tag{30}$$

With these weights, one may approximate the weights of the anti-symmetric 3-point elemental filters. Approximation is necessary because of the irrational number $\sqrt{3}$ appearing in (30). One gets

$$\begin{aligned} a &= 1.227 \\ b &= -\ .366 \\ c &= .139 \end{aligned} \tag{31}$$

The virtue of this filter is that it doesn't yield amplification anywhere in the range $-1 \leq \zeta \leq +1$. It differs in this respect from the filter designed by Shuman, which does admit of very small amplification. The filter response at 4 grid intervals (ζ=0) is 0.875 compared with about 0.95 for the filter discussed by Shuman; at 5 grid intervals, the response is about .96 as contrasted with unity.

4. Concluding Remarks

I was unable to generalize the form of the filter response function when one requires higher-order derivatives of R to vanish at $\zeta = +1$ in addition to the other conditions of the H-filter.

The elimination of any need to do complex arithmetic when using 3-point elemental filters opens the way for a careful analysis of the end conditions' influence when elemental filters are used recursively.

I have coded a two-dimensional application of the filter whose response function is given in (24). The code is attached. When KINDEX = 3, this is a 9 x 9 operator that I will call the 'N-filter.'

```
OPTIONS: NODECK,NOLIST,OPTIMIZE(0),NOMAP,SIZE(MAX),NOIL,NOXREF,NOTERM,LC(6
N EFFECT: NAME(MAIN) NOOPTIMIZE LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
          SOURCE EBCDIC NOLIST NODECK OBJECT NOMAP NOFORMAT GOSTMT NOXREF A

             FUNCTIONS INLINE ARE:    NONE

02           SUBROUTINE FILTER(F,W,N,M,KINDEX)
03           DIMENSION F(N,M), W(N,M)
       C       THIS SUBROUTINE PERFORMS A 2 DIMENSIONAL FILTERING OF THE
       C       FIELD F,WITH DIMENSIONS M*N.  THE FILTER USED DEPENDS
       C      UPON KINDEX
       C        KINDEX=1  IS THE H FILTER
       C      KINDEX = 2 IS THE G FILTER
       C      KINDEX = 3 IS THE N FILTER
       C       FOR REFERENCE (CF. GERRITY OFFICE NOTE 169  , 1978)
       C       W IS A WORK FIELD WITH DIMENSIONS N*M
       C       THE SMOOTHED FIELD OVERWRITES THE INPUT FIELD F
04             NN = N-1
05             MM = M-1
06             DO 10 K=1,KINDEX
07               DO 20 J=1,M
08             DO 30 I=2,NN
09           W(I,J) = .25*(F(I-1,J)+F(I+1,J) ) + .5 * F(I,J)
10        30 CONTINUE
11           W(1,J) = F(1,J)
12           W(N,J) = F(N,J)
13        20 CONTINUE
14             DO 40 I=1,N
15             DO 50 J=2,MM
16             F(I,J) = .25*(W(I,J+1)+W(I,J-1)) + .5*W(I,J)
17        50 CONTINUE
18           F(I,1) = W(I,1)
19             F(I,M) = W(I,M)
20        40 CONTINUE
21        10 CONTINUE
       C
22             XNU = -.5*FLOAT(KINDEX)
23           XEND = .5*XNU
24           XMID= 1. -2.*XEND
25             DO 60 J=1,M
26             DO 70 I=2,NN
27             W(I,J) = XEND*(F(I+1,J)+F(I-1,J)) + XMID*F(I,J)
28        70 CONTINUE
29             W(1,J) = F(1,J)
30             W(N,J) = F(N,J)
31        60 CONTINUE
32               DO 80 I=1,N
33           DO 90 J=2,MM
34                F(I,J) = XEND*(W(I,J+1)+W(I,J-1)) +XMID*W(I,J)
35        90 CONTINUE
36             F(I,1) = W(I,1)
37              F(I,M) = W(I,M)
38        80 CONTINUE
39             RETURN
40             END

IN EFFECT*NAME(MAIN) NOOPTIMIZE LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
```