U. S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE
NATIONAL METEOROLOGICAL CENTER

OFFICE NOTE 156

The LFM-II Time Integration Method

Joseph P. Gerrity, Jr.
Development Division

AUGUST 1977

This is an unreviewed manuscript, primarily intended for informal exchange of information among NMC staff members.

The LFM-II Time Integration Method

1. Introduction

The improvement of numerical weather prediction through the reduction of space-derivative truncation error is an important objective. The most straightforward approach to obtaining improved accuracy is through the reduction of the horizontal grid mesh. Reduction of the horizontal grid mesh requires a related reduction of the integration time step to maintain computational stability.

The LFM-II model has been designed to use a horizontal grid interval just two-thirds the size of the value used in the current LFM model. If the numerical method for marching the calculation forward in time was left unchanged from the method used in the current LFM model (cf. Gerrity and Newell, 1976), the time step would have to be reduced to two-thirds of the presently employed value. Since the area covered by the LFM-II model is identical to the area currently covered by the LFM model, the amount of computing required by the LFM-II could have increased to $(3/2)^3$, i.e., by approximately the factor 3.4, of the time currently required.

In order to reduce the requirement for additional computing, it was decided to change the time-marching procedure. The new method involves the use of a weighted time average of the pressure gradient in the momentum equations, and the use of a time filter on the dependent variables. A linear analysis of the new method, omitting advection and diffusion terms, has been presented by Brown and Campana, 1977. The influence of advection was analyzed by Schoenstadt and Williams (1976). In this note, we consider the impact of the incorporation of spatial diffusion.

2. The Model Equations

We consider the following model equations

$$\frac{\partial u}{\partial t} = - c \frac{\partial p}{\partial x} + K \frac{\partial^2 u}{\partial x^2} , \tag{1a}$$

$$\frac{\partial p}{\partial t} = - c \frac{\partial u}{\partial x} + K \frac{\partial^2 p}{\partial x^2} , \tag{1b}$$

with c and K both positive constants. The dependent variables u and p are assumed to be superpositions of trigonometric functions of the space coordinate. We specify the wave number by k.

The time integration method of the LFM-II model is applied to equations (1). The spatial derivatives are assumed to be centered, second order accurate approximations. We define the following parameters:

α the pressure gradient average weight,

β the Robert time filter weight,

$\hat{\alpha} = 1 - 2\alpha,$

$\hat{\beta} = 1 - 2\beta,$

$\sigma = \frac{K\Delta t}{(\Delta x)^2} 2(1 - \cos k\Delta x),$

$\nu = \frac{c\Delta t}{\Delta x} (\sin k\Delta x),$

$\hat{\sigma} = 1 - 2\sigma .$

The numerical approximation is written as the system of equations,

$$\hat{u}^{n+1} = \hat{\sigma}\, u^{n-1} - 2i\nu(\alpha\, \hat{p}^{n+1} + \alpha\, p^{n-1} + \hat{\alpha}\, \hat{p}^{n}), \tag{2a}$$

$$\hat{p}^{n+1} = \hat{\sigma}\, p^{n-1} - 2i\nu(\hat{u}^{n}), \tag{2b}$$

$$p^{n} = \beta\, p^{n-1} + \beta\, \hat{p}^{n+1} + \hat{\beta}\, \hat{p}^{n}, \tag{2c}$$

$$u^{n} = \beta\, u^{n-1} + \beta\, \hat{u}^{n+1} + \hat{\beta}\, \hat{u}^{n}, \tag{2d}$$

With $\hat{\sigma} = 1$ (i.e., $\sigma = 0$), this system is equivalent to that analyzed by Brown and Campana (1977).

Through algebraic manipulation of the system (2), which holds for all n, greater than n = 1, one may obtain a set of two equations in u and p:

$$A_1\, u^{n+1} + A_2\, u^n + A_3\, u^{n-1} + i(A_4\, p^{n+1} + A_5\, p^n + A_6\, p^{n-1}) = 0 \qquad (3a)$$

$$i(B_1\, u^{n+1} + B_2\, u^n + B_3\, u^{n-1}) + (B_4 p^{n+1} + B_5\, p^n + B_6\, p^{n-1}) = 0 \qquad (3b)$$

The parameters A and B are related to α β σ and ν via the equations (4) below.

$$A_1 = \hat{\beta} \qquad (4a)$$

$$A_2 = 4\nu^2\ (\alpha - \beta) - \beta\hat{\beta}(1 + \hat{\sigma}) \qquad (4b)$$

$$A_3 = -\ (\hat{\sigma}\hat{\beta}^2 + 4\nu^2\beta(\alpha-\beta)) \qquad (4c)$$

$$A_4 = 2\nu\hat{\alpha}\beta \qquad (4d)$$

$$A_5 = 2\nu[\beta(\alpha-\beta)(1+\hat{\sigma}) + \hat{\alpha}\hat{\beta}] \qquad (4e)$$

$$A_6 = 2\nu\hat{\beta}(\alpha-\beta)(1+\hat{\sigma}) \qquad (4f)$$

$$B_1 = 2\nu\beta \qquad (4g)$$

$$B_2 = 2\nu[\hat{\beta} - \beta^2(1+\hat{\sigma})] \qquad (4h)$$

$$B_3 = -\ 2\nu\beta\hat{\beta}(1+\hat{\sigma}) \qquad (4i)$$

$$B_4 = \hat{\beta} - 4\nu^2\hat{\alpha}\beta \qquad (4j)$$

$$B_5 = -\ [\beta\hat{\beta}(1+\hat{\sigma}) + 4\nu^2\hat{\alpha}\beta] \qquad (4k)$$

$$B_6 = -\ [\hat{\sigma}\hat{\beta}^2 + 4\nu^2\beta(\alpha-\beta)] \qquad (4\ell)$$

By requiring the dependent variables to depend upon n via

$$\begin{pmatrix} u^n \\ p^n \end{pmatrix} = \lambda^n \begin{pmatrix} U_o \\ P_o \end{pmatrix} \qquad (5)$$

one obtains the characteristic polynomial in λ

$$(A_1\lambda^2 + A_2\lambda + A_3)(B_4\lambda^2 + B_5\lambda + B_6) + (B_1\lambda^2 + B_2\lambda + B_3)(A_4\lambda^2 + A_5\lambda + A_6) = 0 \tag{6}$$

The coefficient of λ^4 is

$$\hat{c}_o = A_1B_4 + A_4B_1 = \hat{\beta}^2 - 4\nu^2\beta(\alpha-\beta) \tag{7}$$

Provided that $\hat{c}_o$ is not zero we may express the polynomial (6) as

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0. \tag{8}$$

The coefficients can be expressed,

$$c_1 = 4\nu^2\alpha - 2(1+\hat{\sigma})\beta, \tag{9A}$$

$$c_2 = \hat{c}_o^{-1}\{\hat{\beta}^2[(1+\hat{\sigma})^2\beta^2 - 2\hat{\sigma}\beta + 4\nu^2(\hat{\alpha}-\alpha\beta)] - 4\nu^2\beta(\alpha-\beta)[(1+\hat{\sigma})^2\beta^2 + (\hat{\sigma}-1) - (3\hat{\sigma}-1)\beta + 4\nu^2(\hat{\alpha}-\alpha\beta)]\} \tag{9B}$$

$$c_3 = 2\hat{\sigma}(1+\hat{\sigma})\beta\hat{\beta} + 4\nu^2(\alpha-\beta-\hat{\alpha}\beta), \tag{9C}$$

$$c_4 = \hat{\sigma}^2\hat{\beta}^2 - 4\nu^2\beta(\alpha-\beta). \tag{9D}$$

When no space-diffusion-term is used, $\hat{\sigma} = 1$, and the coefficients (9) reduce to the form given as eq. (17) by Brown and Campana.

The stability of the LFM-II time integration method (including space smoothing) can be estimated by solving eq. (8) for the roots of the polynomial. Since the coefficients c are all real, one may develop the roots of (8) by the use of radicals. This will be done in section (3). Alternatively, one may solve for the roots numerically using a set of values for the parameters α, β, σ and ν. This is done in section (4).

3. Solution by Radicals

Given the equation

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0 \qquad (10)$$

with real coefficients, one may use the transformation

$$x = \lambda + \tfrac{1}{4}c_1$$

to reduce (10) to the form,

$$x^4 + p\,x^2 + qx + r = 0 \qquad (11)$$

with

$$p = c_2 - 3\,c_1^2/8 \qquad (12A)$$

$$q = c_3 - c_1(c_1/4)^2 - c_1 p/2 \qquad (12B)$$

$$r = c_4 - (c_1/4)^2[(c_1/4)^2 + p] - c_1 q/4 \qquad (12C)$$

If $q = 0$, one may solve (11) directly

$$x^2 = -\frac{p}{2} \pm \tfrac{1}{2}(p^2 - 4r)^{\frac{1}{2}} \qquad (13)$$

If $p^2 - 4r$ is non-negative,

$$x_{1,2} = \pm\left[-\frac{p}{2} + \tfrac{1}{2}(p^2 - 4r)^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (14A)$$

$$x_{3,4} = \pm\left[-\frac{p}{2} - \tfrac{1}{2}(p^2 - 4r)^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (14B)$$

If $p^2 - 4r$ is negative, $r > 0$ and

$$x^2 = -\frac{p}{2} \pm \frac{i}{2}(4r - p^2)^{\frac{1}{2}}\ . \qquad (15)$$

Let

$$\mu \equiv \tan^{-1}\left(-\frac{(4r - p^2)^{\frac{1}{2}}}{p}\right) \qquad (16A)$$

One may then solve the two forms of (15),

$$x^2 = r\, e^{i\mu} \qquad\qquad x^2 = r\, e^{-i\mu}$$

$$x_1 = r^{\frac{1}{2}}\, e^{i\frac{1}{2}\mu} \qquad\qquad x_3 = r^{\frac{1}{2}}\, e^{-i\frac{1}{2}\mu} \tag{17A,B}$$

$$x_2 = r^{\frac{1}{2}}\, e^{i(\frac{1}{2}\mu+\pi)} \qquad\qquad x_4 = r^{\frac{1}{2}}\, e^{-i(\frac{1}{2}\mu+\pi)} \tag{17C,D}$$

Thus if $q = 0$ and $p^2 - 4r \geq 0$, the roots are

$$\lambda_1 = -\tfrac{1}{4}c_1 + [-\tfrac{p}{2} + \tfrac{1}{2}(p^2 - 4r)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{18A}$$

$$\lambda_2 = -\tfrac{1}{4}c_1 - [-\tfrac{p}{2} + \tfrac{1}{2}(p^2 - 4r)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{18B}$$

$$\lambda_3 = -\tfrac{1}{4}c_1 + [-\tfrac{p}{2} - \tfrac{1}{2}(p^2 - 4r)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{18C}$$

$$\lambda_4 = -\tfrac{1}{4}c_1 - [-\tfrac{p}{2} - \tfrac{1}{2}(p^2 - 4r)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{18D}$$

If $q = 0$ and $p^2 - 4r < 0$, $(r>0)$, then the roots are,

$$\lambda_1 = -\tfrac{1}{4}c_1 + r^{\frac{1}{2}}\, e^{i\frac{1}{2}\mu} \tag{19A}$$

$$\lambda_2 = -\tfrac{1}{4}c_1 + r^{\frac{1}{2}}\, e^{i(\frac{1}{2}\mu+\pi)} \tag{19B}$$

$$\lambda_3 = -\tfrac{1}{4}c_1 + r^{\frac{1}{2}}\, e^{-i\frac{1}{2}\mu} \tag{19C}$$

$$\lambda_4 = -\tfrac{1}{4}c_1 + r^{\frac{1}{2}}\, e^{-i(\frac{1}{2}\mu+\pi)} \tag{19D}$$

If $q \neq 0$, one may find u, a real number greater than p, such that eq. (11) may be re-expressed as

$$(x^2 + Ax + \tfrac{u}{2} - B)(x^2 - Ax + \tfrac{u}{2} + B) = 0 \tag{20}$$

with $\quad A = (u - p)^{\frac{1}{2}}$, $\hfill$ (21A)

$\quad B = q/(2A)$. $\hfill$ (21B)

The parameter u must be selected so that it satisfies the cubic,

$$f(u) = (u - p)(u^2 - 4r) - q^2 = 0 \qquad (22)$$

Clearly, for q nonzero, f(u) is negative when $u = p$ and becomes positive for sufficiently large values of u; consequently, a root exists for some $u > p$.

Since the coefficients of (22) are real, one may find at least one real root of (22). By use of the transformation,

$$y = u - \tfrac{1}{3}p \qquad (23)$$

eq. (22) becomes

$$y^3 + sy = t \qquad (24)$$

with

$$s = -\left[4r + \tfrac{1}{3}p^2\right] \qquad (25A)$$

and

$$t = q^2 - 4rp - \left(\tfrac{1}{3}p\right)^3 - \tfrac{1}{3}\,sp\ . \qquad (25B)$$

The further transformation

$$y = \sqrt{\tfrac{4}{3}|s|}\; z \qquad (26)$$

yields

$$4z^3 + 3\,\frac{s}{|s|}\,z = C \qquad (27)$$

with

$$C = \frac{1}{2}\left(\frac{3}{|s|}\right)^{3/2} \qquad (28)$$

Equation (27) is amenable to solution through recourse to specific trigonometric identies, viz.,

$$\sinh 3\theta = 4(\sinh\theta)^3 + 3(\sinh\theta), \qquad (29A)$$

$$\cosh 3\theta = 4(\cosh\theta)^3 - 3(\cosh\theta), \qquad (29B)$$

$$\cos 3\theta = 4(\cos\theta)^3 - 3(\cos\theta). \qquad (29C)$$

The appropriate choice among eqs. (29) depends upon the sign of s and the magnitude of C.

If $s > 0$, (29A) is appropriate.

If $s > 0$, and $|C| < 1$, one uses (29C), otherwise (29B) is appropriate.

Should $s = 0$, the solution of (27) is immediate.

Having obtained the root for z, back-substitution yields a real root u_1 of (22). If the u root exceeds p, it may be used directly in (20) and (21). Should the root obtained for u_1 not exceed p, one may divide the linear factor $(u-u_1)$ out of (22) and find the remaining two necessarily real roots of (22). One of these must exceed p, as we have shown earlier, and it may be used in (20) and (21).

4. Numerical Evaluation of the Roots

As suggested by the complexity of the analytic method for extracting the roots of the characteristic polynomial, outlined in section 3, the most tractable approach to studying the stability of the LFM-II integration system is through numerical evaluation of the roots. Reasonable choices of the parameters of the problem are suggested by the experience previously gained with the component parts of the new integration method.

With respect to the use of diffusion in the LFM, we found[1] that the stability criterion,

$$\Delta t < \sqrt{\hat{\sigma}} \cdot 424 \text{ sec.} \qquad (30)$$

agreed well with empirical experience. In developing (30) we used the value $\sqrt{2}$ (120 km) for the grid interval Δx. The $\sqrt{2}$ appears because of the

[1] Office Note 129.

lattice structure of the semi-momentum difference system. The 120 km parameter represents the minimum grid interval (at about 15°N) of the LFM model (Δx = 190.5 km at 60°N on the map projection). The phase speed of the fastest wave was set at 400 m sec^{-1}, which is a slightly conservative choice that involves the addition of a mean flow (about 100 m sec^{-1}) to the gravity wave speed (about 300 m sec^{-1}).

The value of the parameter $\hat{\sigma}$ used in the LFM (β = .1 in Office Note 129) is approximately .84 for $k\Delta x = \pi$. The equivalent diffusion coefficient K would have to be a function of wave number. For $k\Delta x = \pi/2$ a reasonable estimate is $2.5 \times 10^5 m^2 sec^{-1}$, but at $k\Delta x = \pi$ the coefficient would be an order of magnitude larger. We may compromise with K equal to $5 \times 10^5 m^2 sec^{-1}$.

The use of the Robert time filter was analyzed[2] for a wave equation with true frequency ω. We found that a leapfrog scheme must use a time step maximum set by the parameter $\hat{\beta}$,[3] viz.,

$$\nu = \omega \Delta t \left(\frac{1+\hat{\beta}}{3-\hat{\beta}}\right)^{1/2} \tag{31}$$

For a diffusion equation, the Robert time filter was found[4] to pose no greater restriction on the time step than is ordinarily the case. For a choice of $\beta = .025$, $\hat{\beta} = .95$ and one finds $\nu < .98$, which is not a serious restriction.

---

[2]Office Note 62

[3]In the cited office note, $\hat{\beta}$ was denoted by α.

[4]Office Note 60

When diffusion was not involved, Brown and Campana (op. cit.) found that an optimal choice of α could be made for a given value of β, viz.,

$$\alpha_{opt} = (\beta^2 + 1)(\beta + 1)/4. \tag{32}$$

Thus for the LFM we might expect the maximum time step allowed to be

$$\Delta t_{max} = (.98)\sqrt{.84}\ (424) \text{ secs} \tag{33}$$

if α is zero. If by the selection of an optimal α value from (32) we can obtain the suggested sixty percent increase in Δt*, then the LFM-II method applied to the LFM should be stable with

$$\Delta t = 1.6\ \Delta t_{max} = 609 \text{ secs}. \tag{34}$$

For the LFM-II grid, which is 2/3 the size of the LFM's grid interval, the factor 424 secs is reduced to 283 secs, and thus one could hope to obtain stability with

$$\Delta t = 1.6\ [.98\sqrt{.84}\ \ 283] = 407 \text{ secs}. \tag{35}$$

With this as background, we may now discuss the results of numerical solutions of this characteristic polynomial.

4.1 Interpretation of Numerical Evaluation of the Roots

Our interest centers about the stability of the integration method rather than details of the linear solution--this is the case because the accuracy of the numerical weather predictions depends upon accurate non-linear effect simulation, not upon the linear effects. Consequently, we

---

*cf. p. 9 of Brown and Campana (op.cit.)

will examine only the root with maximum absolute value as the parameters of the problem are varied. The roots of the characteristic polynomial depend upon four nondimensional numbers: α, β, σ, and ν. Our specific interest lies in the dependence upon σ (or $\hat{\sigma}$).

The simplest course of action was to evaluate the largest (magnitude) root for fixed α and β, as $\hat{\sigma}$ and ν vary. We selected three values of β and calculated α from β using the formula suggested by Brown and Campana. The three values of β are 0.025, 0.050, and 0.075. The results of these computations are shown in figures 1, 2, and 3.

The results displayed in Figures 1, 2, and 3 show that the presence of the spatial diffusion term restricts the maximum value of ν for which the integration will be stable. There are only small differences among the results as β and α vary.

Figure 4 is a diagram showing the maximum root for $\alpha = .25$, $\beta = 0$, Figure 5 is a similar diagram for $\alpha = 0$, $\beta = .025$.

In Figure 6 we bring together in one diagram the neutral ($|\lambda| = 1$) curves for several choices of α and β, labeled A, B, C, and D.

Comparison of curves A and B shows that the use of the time filter ($\beta = .025$) reduces the maximum value of ν below unity for $\hat{\sigma} = 1$ (no diffusion). As $\hat{\sigma}$ decreases (larger diffusion) the time filter's effect is diminished, until with $\hat{\sigma} \leq .7$, we find its effect is to permit a larger value of ν.

Curves C and D are constructed for a value of approximately .25. Over the entire range of $\hat{\sigma}$, these curves indicate that appreciably greater values of ν may be used stably. These curves also show that for small diffusion ($\hat{\sigma} \approx .9$), the use of the time filter and the related optimal α value permits larger values of ν. Over the bulk of the range of $\hat{\sigma}$ the difference between the C and D curves is nil.

4.2 Implications

We turn now to the implications of the analysis for the practical use of the method in the LFM-II. The basic question is, how large a time step may be used for specific choices of the space diffusion coefficient K?

Let's recall the definitions of $\hat{\sigma}$ and ν.

$$\hat{\sigma} = 1 - \frac{4K\Delta t}{(\Delta x)^2} (1 - \cos k\Delta x)$$

$$\nu = \frac{c\Delta t}{\Delta x} \sin k\Delta x \ .$$

The parameter $k\Delta x$ ranges between 0 and π. The factor $(1 - \cos k\Delta x)$ takes its largest value when $k\Delta x$ is π. The factor, $\sin k\Delta x$, takes its largest value when $k\Delta x$ is $\pi/2$.

In the LFM-II model the smallest value of Δx is 80 km, and since the vertical structure is similar to the 6L PE, the external gravity wave (Lamb wave) will have a phase speed c of 330 m sec^{-1} for the U.S. Standard Atmosphere.*

---

*cf. NMC Office Note 74.

The diffusion coefficient K has been assigned values between 10^5 and 6.10^5 m^2sec^{-1}, and experimental runs have been performed with Δt ranging over 360 and 400 secs. In these runs β = .075 and α = .2703.

Since the truncation error factors vary between diffusion and wave motion terms, we tabulate below the values of $\hat{\sigma}$ and ν for various values of $k\Delta x$ when different Δt and K values are used.

Case	Δt sec	K $\frac{m^2}{sec}$	c $\frac{m}{sec}$	Δx m
A	360	10^5	330	8.10^4
B	360	6.10^5	330	8.10^4
C	372	10^5	330	8.10^4
D	372	6.10^5	330	8.10^4
E	400	10^5	330	8.10^4
F	400	6.10^5	330	8.10^4

Case	$k\Delta x$	$.5\pi$	$.6\pi$	$.7\pi$	$.8\pi$	$.9\pi$
A		1.48	1.41	1.20	.87	.46
B		1.48	1.41	1.20	.87	.46
C		1.53	1.46	1.24	.90	.47
D		1.53*	1.46	1.24	.90	.47
E		1.65	1.57	1.33	.97	.51
F		1.65*	1.57*	1.33	.97	.51

ν values

Case	kΔx	.5π	.6π	.7π	.8π	.9π
A		.978	.971	.964	.959	.956
B		.865	.823	.786	.756	.737
C		.977	.970	.963	.958	.955
D		.860*	.817	.778	.748	.728
E		.975	.967	.960	.955	.951
F		.850*	.804*	.762	.729	.707

$\hat{\sigma}$ values

The points $(\hat{\sigma},\nu)$ for the several cases tabulated have been checked with figure 3. The asterisks in the tables show those instances in which the diagram indicates numerical instability.

We conclude that for $K = 6.10^6$ the time step Δt must be less than 372 secs. For $K = 10^5$, the time step may be taken as large as 400 secs.

5.0 Summary

We have shown that the numerical method proposed for use in the LFM-II model will allow the use of a relatively longer time step than the conventional leapfrog method. We also found that the includion of a space diffusion term restricts the size of the time step. For a choice of $K = 6.10^5 m^2 sec^{-1}$, one would have to use a time step less than 372 secs in order to satisfy the constraint developed in this analysis.

It is fairly clear that the precise form of the diffusion term is of some importance in optimizing the time step. It would be instructive to consider the result of using a nonlinear diffusion term. Very large diffusion (small $\hat{\sigma}$) is likely to be permitted for small wave lengths (ν small).

It should also be noted that more accurate estimation of ν, say through the use of fourth order differences, will require a more selective diffusion operator.

6.0 References

Schoenstadt, A. L., and R. T. Williams, 1976: The Computational Stability Properties of the Shuman Pressure Gradient Averaging Technique. Journal of Computational Physics, 21, 166-177.

Brown, J. A., and K. A. Campana, 1977: An Economical Time-Differencing System for Numerical Weather Prediction. (Manuscript submitted for publication.) NMC

Gerrity, J. P., and J. E. Newell, 1976: A Note on the LFM Time Integration, NMC Office Note 129.

Gerrity, J. P., and S. H. Scolnik, 1971: Some Comments on Robert's Time Filter for Time Integration, NMC Office Note 62.

Gerrity, J. P., and S. H. Scolnik, 1971: Further Properties of Time Averaging as Applied to Wave Type and Damping Type Equations, NMC Office Note 60.