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On Map Projections for Numerical Weather Prediction

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1. Introduction

Standard meteorological texts usually discuss the equations governing the atmosphere in either spherical coordinates or on a β plane. In the practice of numerical weather prediction, the equations are usually transformed into the map coordinates of a conformal map. This note is an attempt to bring together in one convenient place the definitions of the maps used and to indicate the methodology by which the transformation of the equations may be carried through.

It has become evident that the standard polar stereographic map which has served NWP for so long is no longer an appropriate projection. Attention is shifting toward global and limited area models, for neither is the usage of the polar stereographic map optimal.

The sources of material used in this note are Godske, Bergeron, Bjerknes and Bundgaard (1957), and Deetz and Adams (1945).

2. The Lambert Conformal Map

The surface spherical coordinate λ, longitude, and ϕ, latitude, may be mapped into the plane polar coordinates, r, radial distance, and θ, azimuthal angle by the transformation

$$r = r_o \left[\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right]^K \qquad \text{1a}$$

$$\theta = K(\lambda - \lambda_o) \qquad \text{1b}$$

in which r_o and K are positive, real valued constants and λ_o is an arbitrary zero reference for longitude.

If we take $K \leqq 1$, the pole $\phi = \pi/2$ maps into $r = 0$ and the pole $\phi = -\pi/2$ maps into the point at infinity. The mapping is not "one to one"; since $K < 1$, the points in the sector, $2\pi K < \theta + \lambda_o < 2\pi$, do not correspond to points on the sphere.

The mapping is conformal since

$$\frac{1}{a\cos\phi}\frac{\partial r}{\partial \lambda} = +\frac{r}{a}\frac{\partial \theta}{\partial \phi} = 0 \qquad \text{2a}$$

$$\frac{r}{a\cos\phi}\frac{\partial \theta}{\partial \lambda} = \frac{-1}{a}\frac{\partial r}{\partial \phi} = \frac{rK}{\cos\phi} \qquad \text{2b}$$

in which a is the radius of the spherical surface.

Since there are two parameters r_o and K at our disposal, one may set two compatible constraints upon the mapping. Typically, one may set the scale on the map equal to that on the spherical surface at two latitudes, ϕ_1 and ϕ_2. One requires

$$r\,d\theta = a\cos\phi_1\,d\lambda \qquad \text{at } \phi = \phi_1 \qquad \text{3a}$$

$$r\,d\theta = a\cos\phi_2\,d\lambda \qquad \text{at } \phi = \phi_2 \qquad \text{3b}$$

These equations give

$$r_o K \left(\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\right)^K = a\cos\phi_1 \qquad \text{4a}$$

$$r_o K \left(\tan\left(\frac{\pi}{4} - \frac{\phi_2}{2}\right)\right)^K = a\cos\phi_2 \qquad \text{4b}$$

One may solve these for K and r_o to get,

$$K = \ell n\left(\frac{\cos\phi_1}{\cos\phi_2}\right) \div \ell n\left(\frac{\tan\frac{\pi}{4} - \frac{\phi_1}{2}}{\tan\frac{\pi}{4} - \frac{\phi_2}{2}}\right) \qquad \text{5}$$

and

$$r_o = \frac{a\cos\phi_1}{K}\left(\tan\frac{\pi}{4} - \frac{\phi_1}{2}\right)^{-K} \qquad \text{6}$$

The map scale factor, m, at an arbitrary point on the map is defined as the ratio of distance on the map to distance on the sphere. Measuring the distances along the parallel of latitude ϕ, one finds

$$m(\phi) = \frac{r\,d\theta}{a\cos\phi d\lambda} = \frac{\cos\phi_1}{\cos\phi}\left(\frac{\tan\frac{\pi}{4} - \frac{\phi}{2}}{\tan\frac{\pi}{4} - \frac{\phi_1}{2}}\right)^K \qquad \text{7a}$$

or equivalently,

$$m(\phi) = \left(\frac{\cos\phi_1}{\cos\phi}\right)^{1-K}\left(\frac{1+\sin\phi_1}{1+\sin\phi}\right)^K \qquad \text{7b}$$

In terms of m as defined in eq. 7 and of the results obtained in eqs. 5 and 6, one may rewrite eqs. 1 as

$$r = a\ m(\phi)\ \cos\phi/K \tag{8a}$$

$$\theta = K(\lambda - \lambda_o) \tag{8b}$$

In terms of Cartesian coordinates, (x,y), on the plane map, one has

$$x = r\cos\theta = a\ m(\phi)\ \cos\phi\ \cos[K(\lambda - \lambda_o)]/K \tag{9a}$$

$$y = r\sin\theta = a\ m(\phi)\ \cos\phi\ \sin[K(\lambda - \lambda_o)]/K \tag{9b}$$

The metrics, h_x and h_y, for the coordinate system x,y, are calculable (cf. Morse and Fishbach, p. 24) from the relationships

$$1 = h_x^2 \left\{\left(\frac{1}{a\cos\phi}\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{1}{a}\frac{\partial x}{\partial\phi}\right)^2\right\} \tag{10a}$$

$$1 = h_y^2 \left\{\left(\frac{1}{a\cos\phi}\frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{1}{a}\frac{\partial y}{\partial\phi}\right)^2\right\} \tag{10b}$$

One may prove that

$$\frac{\partial}{\partial\phi}\left\{m(\phi)\ \cos\phi\right\} = -\ K\ m(\phi) \tag{11}$$

Evaluation of the partial derivatives in eqs. 10 yields

$$h_x = h_y = \frac{1}{m(\phi)} \tag{12}$$

2.1 The Quasi-Static Equations in Map Coordinates

The equations of motion in surface spherical coordinates can be written in the following form, which is compatible with the quasi-static approximation and the conservation of total energy (Lorenz, 1967, p. 18),

$$\frac{du_s}{dt} - \frac{u_s v_s \sin\phi}{a\cos\phi} - fv_s + \frac{\alpha}{a\cos\phi}\frac{\partial p}{\partial\lambda} = F_\lambda \tag{13a}$$

$$\frac{dv_s}{dt} + \frac{u_s u_s \sin\phi}{a\cos\phi} + fu_s + \frac{\alpha}{a}\frac{\partial p}{\partial\phi} = F_\phi \tag{13b}$$

in which $u_s = a\cos\phi\dot{\lambda}$ and $v_s = a\dot{\phi}$

If we define the map wind components

$$u = h_x \frac{dx}{dt} = \frac{1}{m}\dot{x} \qquad 14a$$

$$v = h_y \frac{dy}{dt} = \frac{1}{m}\dot{y} \qquad 14b$$

then the following relationships hold between (u_s, v_s) and (u, v), and, for that matter, among the components of any "horizontal" vectors in the two coordinate systems.

$$u = -v_s \cos K(\lambda - \lambda_o) - u_s \sin K(\lambda - \lambda_o) \qquad 15a$$

$$v = -v_s \sin K(\lambda - \lambda_o) + u_s \cos K(\lambda - \lambda_o) \qquad 15b$$

To derive equations in the map coordinate, eqs. 13 are multiplied by appropriate combinations of $\sin K(\lambda - \lambda_o)$ and $\cos K(\lambda - \lambda_o)$, and are then added. The observations that

$$\vec{K}\cdot\vec{\nabla}x\nabla \ell n\, m \equiv -\frac{u_s(1-\sin\phi)}{a\cos\phi} = v\frac{\partial m}{\partial x} - u\frac{\partial m}{\partial y} \qquad 16a$$

and

$$\frac{1}{a}\frac{\partial}{\partial\phi}\ell n\, m(\phi) = -\frac{1-\sin\phi}{a\cos\phi} \qquad 16b$$

are useful in the manipulation necessary to arrive at,

$$\frac{du}{dt} - \left(f + v\frac{\partial m}{\partial x} - u\frac{\partial m}{\partial y}\right)v + \alpha m\frac{\partial p}{\partial x} = F_x \qquad 17a$$

$$\frac{dv}{dt} + \left(f + v\frac{\partial m}{\partial x} - u\frac{\partial m}{\partial y}\right)u + \alpha m\frac{\partial p}{\partial y} = F_y \qquad 17b$$

The individual derivative may be expressed as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + m(\phi)\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) + w\frac{\partial}{\partial z} \qquad 18$$

The horizontal divergence and the vertical component of the curl of a vector $\vec{v}$ with components (u,v) are given by,

$$\nabla\cdot\vec{v} = m^2(\phi)\left(\frac{\partial}{\partial x}\frac{u}{m} + \frac{\partial}{\partial y}\frac{v}{m}\right) \qquad 19a$$

$$\vec{K}\cdot\nabla x\vec{v} = m^2(\phi)\left\{\frac{\partial}{\partial x}\frac{v}{m} - \frac{\partial}{\partial y}\frac{u}{m}\right\} \qquad 19b$$

In order to express the Coriolis parameter, f, and the map factor, m, in terms of the coordinates (x,y) of a point on the map, one may define

$$R \equiv \left[\frac{K^2(x^2 + y^2)}{a^2\mu^2}\right]^{1/K} \qquad 20$$

with

$$\mu \equiv (\cos\phi_1)^{1-K} \ (1 + \sin\phi_1)^K \qquad 21$$

and then prove that

$$\sin\phi = \frac{1-R}{1+R} \qquad 22$$

and

$$\cos\phi = \left(\frac{2}{1+R}\right)R^{\frac{1}{2}} \qquad 23$$

It then follows that

$$f = 2\omega\left(\frac{1-R}{1+R}\right) \qquad 24a$$

and

$$m = \frac{\mu(1+R)}{2} R^{\frac{K-1}{2}} \qquad 24b$$

3. The Polar Stereographic Map

A special case of the Lambert conformal map is one in which the transformation is modified to

$$r = r_o\left(\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right) \qquad 25a$$

$$\theta = \lambda \qquad 25b$$

This is now a one-to-one, conformal mapping, but one has only one parameter at one's disposal. The scale may be made "true" at one latitude, say ϕ_o.

$$r d\theta = a \cos\phi d\lambda \quad \text{at } \phi = \phi_o$$

One gets

$$r_o \frac{\cos\phi_o}{1+\sin\phi_o} = a \cos\phi_o$$

so

$$r_o = a(1 + \sin\phi_o) \qquad 26$$

The transformation equations may be rewritten as

$$r = a \cos\phi \frac{1 + \sin\phi_o}{1 + \sin\phi} = a\, m(\phi)\cos\phi \qquad 27a$$

$$\theta = \lambda \qquad 27b$$

in which the map factor $m(\phi)$ has been introduced.

4. The Mercator Map

Another special case of the Lambert type conformal mapping uses Cartesian map coordinates

$$x = K\lambda \qquad 28a$$

$$y = F(\phi) \quad \text{with} \quad y = 0 \quad \text{at } \phi = 0 \qquad 28b$$

To make the map conformal, one must satisfy

$$\frac{1}{a}\frac{\partial y}{\partial \phi} = \frac{1}{a \cos\phi}\frac{\partial x}{\partial \lambda} = \frac{K}{a \cos\phi} \qquad 29$$

The functional form of F which satisfies this equation and the condition $y = 0$ at $\phi = 0$, is

$$F = K \int_o^{\phi} \sec\zeta \, d\zeta = K \ln \tan\left(\frac{\pi}{4} + \frac{\zeta}{2}\right)\Big|_o^{\phi}$$

$$F = K \ln\left(\frac{\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)}{\tan\frac{\pi}{4}}\right) = K \ln \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \qquad 30$$

The parameter K may be chosen to make the map true to scale at a latitude ϕ_o,

$$dx = K d\lambda = a \cos\phi d\lambda \qquad \text{at } \phi = \phi_o \qquad 31a$$

$$K = a \cos\phi_o \qquad 31b$$

Thus the Mercator map, true at $\phi = \pm\, \phi_o$, is given by the transformation

$$x = a \cos\phi_o \lambda \qquad 32a$$

$$y = a \cos\phi_o \ln \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \qquad 32b$$

The map scale factor

$$m(\phi) = \frac{\cos\phi_o}{\cos\phi} \qquad 33$$

may be introduced into 32 to get

$$x = a\ m(\phi)\ \cos\phi\lambda\lambda \qquad 34a$$

$$y = a\ m(\phi)\ \cos\phi\ \ell n\ \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \qquad 34b$$

5. Lambert Osculating Conic Map

A final, conformal conic map of the Lambert type is one which is true at just one latitude, ϕ_o. The transformation equations are

$$r = r_o\left(\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right)^{\sin\phi_o} \qquad 35a$$

$$\theta = \left(\sin\phi_o\right)\left(\lambda - \lambda_o\right) \qquad 35b$$

From the conformality equations, one must have

$$\frac{r}{a\cos\phi}\frac{\partial\theta}{\partial\lambda} = \frac{-1}{a}\frac{\partial r}{\partial\phi} \qquad 36$$

But, $$\frac{-1}{a\ r}\frac{\partial r}{\partial\phi} = \frac{\sin\phi_o}{a\cos\phi}$$

and $$\frac{1}{a\cos\phi}\frac{\partial\theta}{\partial\lambda} = \frac{\sin\phi_o}{a\cos\phi}$$

so conformality is assured. The latitude at which the scale is true is the latitude of osculation, ϕ_o. To prove this, we calculate

$$r\ d\theta = r_o\left(\tan\frac{\pi}{4} - \frac{\phi_\phi}{2}\right)^{\sin\phi_o} \sin\phi_o d\lambda$$

and show that this is equal to $a\cos\phi_o d\lambda$ provided that one sets,

$$r_o = a\tan\left(\frac{\pi}{4} - \frac{\phi_o}{2}\right)^{-\sin\phi_o} \text{ctn}\phi_o \qquad 37$$

One may use the identity

$$\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \equiv \frac{\cos\phi}{1 + \sin\phi}$$

and substitute 37 into 35a to get

$$r = \frac{a \cos\phi}{\sin\phi_o} \frac{\cos\phi_o}{\cos\phi} \left(\frac{\cos\phi}{\cos\phi_o}\right)^{\sin\phi_o} \left(\frac{1 + \sin\phi_o}{1 + \sin\phi}\right)^{\sin\phi_o} \qquad 38$$

or

$$r = \frac{a \cos\phi}{\sin\phi_o} m(\phi) \qquad 39$$

in which

$$m(\phi) = \left(\frac{\cos\phi_o}{\cos\phi}\right)^{1 - \sin\phi_o} \left(\frac{1 + \sin\phi_o}{1 + \sin\phi}\right)^{\sin\phi_o} \qquad 40$$

That $m(\phi)$ is the map scale factor is seen by forming the ratio,

$$\frac{r\ d\theta}{a \cos\phi\ d\lambda} = \frac{a\ m \cos\phi \sin\phi_o\ d\lambda}{a \sin\phi_o \cos\phi\ d\lambda} = m(\phi) \qquad 41$$

6. The Pseudo-Spherical Coordinate Map

The partial differential equations written using surface spherical coordinates may be replaced by a set of finite difference equations. This is done in certain models now under development at NMC (Vanderman, 1972) by the construction of a finite difference gridpoint array with the property that points are equally spaced in both latitude and longitude. If this array of points is plotted on a plane surface, the Cartesian coordinates x, y may be defined by means of the transformation

$$x = a\lambda \qquad 42a$$

$$y = a\phi \qquad 42b$$

Considered as a mapping transformation, eq. 42 is neither conformal nor equivalent. The scale factor is different in the two directions

$$m_x = 1/\cos\phi \qquad 43a$$

$$m_y = 1 \qquad 43b$$

One may rewrite the eqs. 42 as

$$x = a\ m(\phi) \cos\phi\ \lambda \qquad 44a$$

$$y = a\ m(\phi) \cos\phi\ \phi \qquad 44b$$

if $m(\phi) \equiv m_x(\phi) = 1/\cos\phi$ 45

To evaluate the metrics h_x and h_y, one has

$$1 = h_x^2\left[\left(\frac{1}{a\ \cos\phi}\ \frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{1}{a}\ \frac{\partial x}{\partial\phi}\right)^2\right]$$

$$1 = h_y^2\left[\left(\frac{1}{a\ \cos\phi}\ \frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{1}{a}\ \frac{\partial y}{\partial\phi}\right)^2\right]$$

$$h_x = \cos\phi \equiv \frac{1}{m} \qquad 46a$$

$$h_y = 1 \qquad 46b$$

The velocity components on the pseudo-map are

$$u = h_x \frac{dx}{dt} = a\ \cos\phi\dot{\lambda} \equiv u_s \qquad 47a$$

$$v = h_y \frac{dy}{dt} = a\ \dot{\phi} \equiv v_s \qquad 47b$$

The equations of motion transform into

$$\frac{\partial u}{\partial t} + m\ u\ \frac{\partial u}{\partial x} + v\ \frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - v\left(\frac{u}{m}\ \frac{\partial m}{\partial y} + f\right) + m\ \alpha\ \frac{\partial p}{\partial x} = F_x \qquad 48a$$

$$\frac{\partial v}{\partial t} + m\ u\ \frac{\partial v}{\partial x} + v\ \frac{\partial v}{\partial y} + w\ \frac{\partial v}{\partial z} + u\left(\frac{u}{m}\ \frac{\partial m}{\partial y} + f\right) + \alpha\ \frac{\partial p}{\partial y} = F_y \qquad 48b$$

Shuman (1970, p. 569) notes that troublesome pair of metric terms may be coalesced into one, if one introduces the wind functions

$$U = u\ \cos\theta - v\ \sin\theta \qquad 49a$$

$$V = u\ \sin\theta + v\ \cos\theta \qquad 49b$$

in which $\theta = \sin\phi_o\ (\lambda - \lambda_o)$ 50

is the azimuthal coordinate of the Lambert Conic Map (cf. sections 5) osculating the sphere at ϕ_o. With 49 defined, the equations transform to

$$\frac{\partial}{\partial t}U + m\ u\ \frac{\partial U}{\partial x} + v\ \frac{\partial U}{\partial y} + w\ \frac{\partial U}{\partial z} - V\ f + \alpha\ \cos\theta\ m\ \frac{\partial p}{\partial x} - \alpha\ \sin\theta\ \frac{\partial p}{\partial y} = 0 \qquad 50a$$

$$\frac{\partial}{\partial t}V + m\ u\ \frac{\partial V}{\partial x} + v\ \frac{\partial V}{\partial y} + w\ \frac{\partial V}{\partial z} + U\ f + \alpha\ \sin\theta\ m\ \frac{\partial p}{\partial x} + \alpha\ \cos\theta\ \frac{\partial p}{\partial y} = 0 \qquad 50b$$

in which we have omitted the frictional force, F, which transforms as the pressure gradient terms.

Shuman proposes that eqs. 49 be modified at each grid point so that ϕ_0 and λ_0 take on the values at the local grid point. The equations given by Shuman, as the set below his eq. 20, are correct only for a local region about the central point where $\theta \simeq 0$ and $U \simeq u$ and $V \simeq v$.

It will be noted that the two poles become lines under this pseudo mapping. The process for treating these singular points may be considered a form of boundary condition specification.

REFERENCES

Deetz, C. H., and O. S. Adams, Elements of Map Projection, U.S. Dept. of Commerce Government Printing Office, Washington, D.C., 1945, 226 pp.

Godske, C. L., T. Bergeron, J. Bjerknes and R. C. Bundgaard, Dynamic Meteorology and Weather Forecasting, American Meteorological Soc., Boston, 1957, 800 pp.

Lorenz, E. N., The Nature and Theory of the General Circulation of the General Circulation of the Atmosphere, World Meteorological Organization, 1967, 161 pp.

Morse, P. M., and H. Fishbach, Methods of Theoretical Physics, Part I, 1953, McGraw-Hill, N.Y.

Shuman, F. G., "On Certain Truncation Errors Associated with Spherical Coordinates," Journal of Applied Meteorology, 1970, 9: pp. 564-570.

Vanderman, L. W., "Forecasting With a Global, Three-Layer Primitive Equation Model," Monthly Weather Review, 1972, 100: pp. 856-868.