U. S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE

MARCH 1971

RESUSCITATION OF AN INTEGRATION PROCEDURE

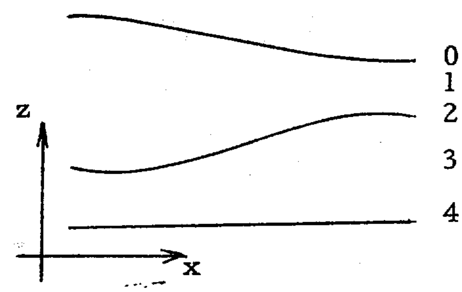
By

F. G. Shuman
National Meteorological Center

OFFICE NOTE 54

Introduction

Let us first look rather precisely at the questions, what is an external, and what is an internal gravity wave. For this purpose we will consider a physical fluid system with only those two modes, consisting of two homogeneous incompressible layers, with the lighter one superposed on the denser.



The numbers to the right of the figure are identifying subscripts, even numbers for surfaces, odd for layers. The system will be regarded as of infinite lateral extent. The perturbation equations for the system may be written

$$\frac{\partial u_1}{\partial t} + g\frac{\partial h_1}{\partial x} + g\frac{\partial h_3}{\partial x} = 0 \tag{1}$$

$$\frac{\partial u_3}{\partial t} + \frac{\rho_1}{\rho_3} g\frac{\partial h_1}{\partial x} + g\frac{\partial h_3}{\partial x} = 0 \tag{2}$$

$$\frac{\partial h_1}{\partial t} + H_1\frac{\partial u_1}{\partial x} = 0 \tag{3}$$

$$\frac{\partial h_3}{\partial t} + H_3\frac{\partial u_3}{\partial x} = 0 \tag{4}$$

where u is velocity, ρ is density and

$$h_1 = z_0 - z_2$$

$$h_3 = z_2 - z_4$$

$$z_4 = 0$$

and H is the time-space mean of h.

We will now look for constants a, b, c_1, c_2 so that the pair of equations

$$\frac{\partial}{\partial t}(a\,u_1 + u_3) + c_1 \frac{\partial}{\partial x}(b\,h_1 + h_3) = 0 \tag{5}$$

$$\frac{\partial}{\partial t}(b\,h_1 + h_3) + c_2 \frac{\partial}{\partial x}(a\,u_1 + u_3) = 0 \tag{6}$$

is consistent with the set (1), (2), (3), (4). These are parametric equations for a wave in the variables $(a\,u_1 + u_3)$ and $(b\,h_1 + h_3)$ with wave speed

$$c = \pm\sqrt{c_1 c_2}$$

The motivation for this approach is that we expect to find a pair of sets of a, b, c, one member of the pair corresponding to the isolated external gravity wave of the original system, the other the internal.

Proceeding, we multiply (1) by a and add to (2), and multiply (3) by b and add to (4).

$$\frac{\partial}{\partial t}(a\,u_1 + u_3) + g\frac{\partial}{\partial x}\left[\left(a + \frac{\rho_1}{\rho_3}\right)h_1 + (a+1)\,h_3\right] = 0 \tag{7}$$

$$\frac{\partial}{\partial t}(b\,h_1 + h_3) + \frac{\partial}{\partial x}\left[b\,H_1\,u_1 + H_3\,u_3\right] = 0 \tag{8}$$

For (5) and (7) to hold simultaneously,

$$c_1\,b = g\left(a + \frac{\rho_1}{\rho_3}\right) \tag{9}$$

$$c_1 = g\,(a+1) \tag{10}$$

and for (6) and (8),

$$c_2\,a = b\,H_1 \tag{11}$$

$$c_2 = H_3 \tag{12}$$

Simultaneous solution of (9), (10), (11), (12) yields

$$a = \frac{H_1 - H_3 \pm \sqrt{(H_1 + H_3)^2 - 4 \frac{\rho_3 - \rho_1}{\rho_3} H_1 H_3}}{2 H_3} \tag{13}$$

$$b = \frac{H_1 - H_3 \pm \sqrt{(H_1 + H_3)^2 - 4 \frac{\rho_3 - \rho_1}{\rho_3} H_1 H_3}}{2 H_1} \tag{14}$$

$$c^2 = c_1 c_2 = g \frac{H_1 + H_3 \pm \sqrt{(H_1 + H_3)^2 - 4 \frac{\rho_3 - \rho_1}{\rho_3} H_1 H_3}}{2} \tag{15}$$

The upper signs correspond to the external gravity wave, for as the density difference $(\rho_3 - \rho_1)$ approaches zero, we get

$$c^2 \cong g (H_1 + H_3) \tag{16a}$$

$$a \cong \frac{H_1}{H_3} \tag{16b}$$

$$b \cong 1 \tag{16c}$$

On the other hand, the lower signs correspond to the internal gravity wave. As the density difference approaches zero we get

$$c^2 \cong 0 \tag{17a}$$

$$a \cong -1 \tag{17b}$$

$$b \cong -\frac{H_3}{H_1} \tag{17c}$$

In the case of the internal gravity wave, we can find how c^2 approaches zero as $(\rho_3 - \rho_1)$ approaches zero by expanding the radical in (15) by means of the binomial series. Then

$$c^2 \cong \frac{\rho_3 - \rho_1}{\rho_3} \frac{g H_1 H_3}{H_1 + H_3} \tag{17d}$$

In summary, (16) suggests that the equations for the external gravity wave may be derived by mass-averaging the equations of motion and summing the h-continuity equations. On the other hand, (17) suggests that the equations for the internal gravity are approximated by differencing the equations of motion and taking the difference of the h_1 -equation weighted by H_3 (the mass of the lower layer) and the h_3 -equation weighted by H_1.

In the work of Gerrity et al. (1971), the equations were not treated completely that way. In their so-called modified semi-implicit method they treated the vertically integrated continuity equation semi-implicitly and the vertically differenced continuity equation explicitly, which indeed corresponds to correct use of the two roots (16c) and (17c) of b.

They treated the equations of motion at each level semi-implicitly, however. My interpretation of the resulting system is that both the equations of motion and mass continuity for the external gravity wave were correctly treated implicitly, but the treatment of the equations for the internal gravity waves was mixed. The equations of motion were treated implicitly, the equations for mass continuity explicitly.

Analysis of "Mixed" Equations

We should be able to investigate the stability characteristics for such a mixed system by analyzing the simplest kind of gravity wave. Consider

$$u_{2t} + g\, \overline{h_{2x}}^{2t} = 0 \tag{18a}$$

$$h_{2t} + H_0\, u_{2x} = 0 \tag{18b}$$

$$u = U \exp i\,(q\,t + r\,x) \tag{19a}$$

$$h = H \exp i\,(q\,t + r\,x) \tag{19b}$$

The operators $(\ )_{2x}$ and $(\ )_{2t}$ are the usual centered difference approximations to the x- and t- derivatives, respectively. The operator $\overline{(\ )}^{2t}$ is a simple centered average at points two time-steps apart. The frequency equation for this system is

$$\sin^2 q\,\Delta t - A \cos q\,\Delta t = 0$$

where

$$A = g H_0 \left(\frac{\sin r \Delta x}{\Delta x / \Delta t} \right)^2 \tag{20}$$

and Δx and Δt are the space and time increments, respectively. The frequency equation may also be written

$$\cos^2 q \Delta t + A \cos q \Delta t - 1 = 0$$

or

$$\cos q \Delta t = \frac{-A \pm \sqrt{A^2 + 4}}{2}$$

In the case of the upper sign

$$0 \leq \frac{-A + \sqrt{A^2 + 4}}{2} \leq 1$$

since $A \geq 0$, and stability is therefore indicated. This is the physical mode, and has two roots, since

$$\cos q \Delta t = \cos(-q \Delta t).$$

In the case of the lower sign, representing the computational modes,

$$\frac{-A - \sqrt{A^2 + 4}}{2} \leq -1$$

and q must therefore be complex. To pin down that at least one root is an amplifying one, let

$$q \Delta t = y + i z$$

where $i = \sqrt{-1}$ and y and z are real. Then

$$\cos q \Delta t = \cos y \cosh z - i \sin y \sinh z$$

$$= - \frac{A + \sqrt{A^2 + 4}}{2}$$

The single imaginary term must vanish, so either $y = (2k + 1) \pi$ where k is an integer, or $z = o$. (Since the hyperbolic cosine of a real number cannot be negative, y cannot be an even multiple of π). However, z cannot be zero, for then $\cosh z = 1$ and $\cos y \leq -1$ which cannot be, for y is defined as real. Therefore, $y = (2k+1)\pi$ and

$$\cosh z = \frac{A + \sqrt{A^2+4}}{2}$$

There are two roots of z, since $\cosh z = \cosh(-z)$. The solution may be written

$$z = \cosh^{-1} w = \pm \ln (w + \sqrt{w^2 - 1})$$

where

$$w = \frac{A + \sqrt{A^2+4}}{2}$$

The frequency then is

$$q = \frac{y + iz}{\Delta t} = \frac{(2k+1)\pi \pm i \ln (w + \sqrt{w^2 - 1})}{\Delta t}$$

and

$$\begin{aligned} u &= U \exp i(qt + rx) \\ &= U \cdot \exp irx \cdot \exp i(2k+1)\pi \tfrac{t}{\Delta t} \cdot \exp \left[\mp \tfrac{t}{\Delta t} \ln(w + \sqrt{w^2 - 1})\right] \\ &= U \cdot \exp irx \cdot (-1)^{\frac{t}{\Delta t}} \cdot \left(w + \sqrt{w^2-1}\right)^{\mp \frac{t}{\Delta t}} \end{aligned}$$

at grid points. The lower sign on the last exponent represents an amplifying root, since $w + \sqrt{w^2 - 1} \geq 1$.

A Stable Mixed System

In place of (18) consider

$$u_{2t} + g\, \overline{h_{2x}}^{tt} = 0 \tag{21a}$$

$$h_{2t} + H_0\, \overline{u_{2x}}^{tt} = 0 \tag{21b}$$

where the operator $\overline{(\ )}^{tt}$ is the centered time-average with weights $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ at adjacent grid points. Substitution from (19) leads to the frequency equation

$$\sin^2 q\Delta t - A \cos^2 \tfrac{1}{2} q \Delta t = 0$$

or

$$4\sin^2\tfrac{1}{2}q\Delta t\cdot\cos^2\tfrac{1}{2}q\Delta t - A\cos^2\tfrac{1}{2}q\Delta t = 0$$

which can be expressed as the two root equations

$$4\sin^2\tfrac{1}{2}q\Delta t = A$$

$$\cos^2\tfrac{1}{2}q\Delta t = 0$$

The former contains the two frequencies for the physical mode, the latter the computational mode. The two computational modes are identical, i.e.-

$$q\Delta t = (2k+1)\pi$$

which is a pure two-increment period, unconditionally neutrally stable. The frequency equation for the physical mode

$$\sin\tfrac{1}{2}q\Delta t = \pm\tfrac{1}{2}\sqrt{A}$$

also exhibits neutral stability with the condition

$$\sqrt{A} \le 2$$

or, referring to (20),

$$\Delta t \le 2\frac{\Delta x}{\sqrt{gH_0}}$$

This is twice the time step allowed by conventional centered differences. Also note that the system (21) (as well as (18)) requires no integrations of equations corresponding to the Helmholtz equations of Robert's semi-implicit system.

<u>Analysis of a system with two internal modes and the external.</u>

The weights (16b) and (16c) for the two-layer case suggest that the external mode can be isolated approximately by summing the equations of motion and summing the continuity equations, if $H_1 = H_3$. One would expect intuitively that summation would hold up as the principle for isolating approximately the external mode in cases of more than two layers. In the next section we will investigate the nature of the approximation in the simple summation, but in this section we will look at another question.

The weights (17b) and (17c), if $H_1 = H_3$, suggest that the difference between the two equations of motion and the difference between the two equations of continuity are a set in which the internal mode is isolated. The question is, what is the generalization for systems with more than two layers? We know a priori that a system with n layers has n-1 internal modes. How should the equations be weighted to isolate one mode from all others?

We will only set out to get a feeling for the answer, not answer the question in its full generality. For this purpose, consider the equations for a three-layer system, similar to the two-layer system illustrated by the figure on the first page.

$$\frac{\partial u_1}{\partial t} + g\frac{\partial h_1}{\partial x} + g\frac{\partial h_3}{\partial x} + g\frac{\partial h_5}{\partial x} = 0$$

$$\frac{\partial u_3}{\partial t} + \frac{\rho_1}{\rho_3} g\frac{\partial h_1}{\partial x} + g\frac{\partial h_3}{\partial x} + g\frac{\partial h_5}{\partial x} = 0$$

$$\frac{\partial u_5}{\partial t} + \frac{\rho_1}{\rho_5} g\frac{\partial h_1}{\partial x} + \frac{\rho_3}{\rho_5} g\frac{\partial h_3}{\partial x} + g\frac{\partial h_5}{\partial x} = 0$$

$$\frac{\partial h_1}{\partial t} + H\frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial h_3}{\partial t} + H\frac{\partial u_3}{\partial x} = 0$$

$$\frac{\partial h_5}{\partial t} + H\frac{\partial u_5}{\partial x} = 0$$

The symbol H here is the mean for each of h_1, h_3, h_5. For simplicity each layer has the same mean depth.

We take the weighted sum of the three equations of motion, using weights a_1, a_3, a_5; and the weighted sum of the three equations of continuity, using weights b_1, b_3, b_5.

$$\frac{\partial}{\partial t}(a_1 u_1 + a_3 u_3 + a_5 u_5)$$

$$+ g\frac{\partial}{\partial x}\begin{bmatrix}(a_1 + a_3 r_2 + a_5 r_2 r_4) h_1 \\ +(a_1 + a_3 + r_4 a_5) h_3 \\ +(a_1 + a_3 + a_5) h_5\end{bmatrix} = 0$$

$$\frac{\partial}{\partial t}(b_1 h_1 + b_3 h_3 + b_5 h_5) + H\frac{\partial}{\partial x}(b_1 u_1 + b_3 u_3 + b_5 u_5) = 0$$

where $\quad r_2 \equiv \dfrac{\rho_1}{\rho_3}$

$\qquad\quad r_4 \equiv \dfrac{\rho_3}{\rho_5}$

We will choose the a's and b's so that these equations are identical to the system

$$\frac{\partial}{\partial t}(a_1 u_1 + a_3 u_3 + a_5 u_5) + c_1\frac{\partial}{\partial x}(b_1 h_1 + b_3 h_3 + b_5 h_5) = 0$$

$$\frac{\partial}{\partial t}(b_1 h_1 + b_3 h_3 + b_5 h_5) + c_2\frac{\partial}{\partial x}(a_1 u_1 + a_3 u_3 + a_5 u_5) = 0$$

Thus

$$c_1 b_1 = g(a_1 + a_3 r_2 + a_5 r_2 r_4)$$

$$c_1 b_3 = g(a_1 + a_3 + r_4 a_5)$$

$$c_1 b_5 = g(a_1 + a_3 + a_5)$$

$$c_2 a_1 = b_1 H$$

$$c_2 a_3 = b_3 H$$

$$c_2 a_5 = b_5 H$$

Eliminating b_1, b_3, b_5, we have

$$\begin{aligned}(c^2 - gH) a_1 - r_2 gH a_3 - r_2 r_4 gH a_5 &= 0\\ -gH a_1 + (c^2 - gH) a_3 - r_4 gH a_5 &= 0\\ -gH a_1 - gH a_3 + (c^2 - gH) a_5 &= 0\end{aligned}$$

where $c^2 = c_1 c_2$, and is the square of the wave speed.

This set of simultaneous equations can yield relationships among a_1, a_3, a_5 only if its determinant is zero.

$$0 = \begin{vmatrix} c^2 - gH & -r_2 gH & -r_2 r_4 gH \\ -gH & c^2 - gH & -r_4 gH \\ -gH & -gH & c^2 - gH \end{vmatrix}$$

We thus obtain the frequency equation

$$0 = \left(\frac{c^2}{gH}\right)^3 - 3\left(\frac{c^2}{gH}\right)^2 + [2(\epsilon_2 + \epsilon_4) - \epsilon_2 \epsilon_4]\left(\frac{c^2}{H}\right) - \epsilon_2 \epsilon_4$$

where
$$\epsilon_2 = 1 - r_2$$
$$\epsilon_4 = 1 - r_4$$

We will assume that

$$\epsilon_2 << 1$$
$$\epsilon_4 << 1$$

Now, we note that if we neglect the last two terms, in the frequency equation, which contain coefficients of the order of ϵ or smaller, two of the roots vanish, and the third root is

$$\frac{c_o^2}{gH} = 3$$

which represents the external mode. The two vanishing roots represent the two internal modes. Since they are vanishingly small compared to that for the external mode, we may approximate them by neglecting the cubic term, if they are of the order ϵ. Solving the resulting quadratic equation, we find (neglecting $\epsilon_2 \epsilon_4$ in comparison to $2(\epsilon_2 + \epsilon_4)$ in the linear term)

$$\frac{c^2}{gH} = \frac{1}{3}\left[(\epsilon_2 + \epsilon_4) \pm \sqrt{(\epsilon_2 + \epsilon_4)^2 - 3\epsilon_2 \epsilon_4}\right]$$

and, indeed, we note that these two roots are of order ϵ.

For illustration of weighting patterns and for convenience, first let $\epsilon_2 = \epsilon_4 = \epsilon$. Then

$$\frac{c^2}{gH} = \tfrac{1}{3}(2 \pm 1)\,\epsilon$$

Thus, the two internal modes are represented by

$$\frac{c_1^2}{gH} = \epsilon$$

$$\frac{c_2^2}{gH} = \tfrac{1}{3}\,\epsilon$$

We may now return to our simultaneous set in the a's, make the same order approximations as we did in deriving the wave speeds, and find the weighting pattern indicated in the following table.

	External	Internal #1	Internal #2
a_1	+1	-1	+1
a_3	+1	0	-2
a_5	+1	+1	+1
c^2	$3gH$	ϵgH	$\frac{1}{3}\epsilon gH$

The only point I make here is that weighting patterns other than the two shown for the internal modes do not isolate the two internal modes from each other. If we add the two patterns (-1, 0, +1) and (+1, -2, +1), we get (0, -2, +2); if we subtract we get (-2, +2, 0). Each of these resulting patterns would (approximately) not include the external mode, but they would each contain a mixture of the two internal modes. Thus, if we merely wanted to isolate the external mode for special treatment as with implicit differences, but were willing to treat the internal modes explicitly, this could be accomplished by treating the summed equations implicitly, and explicitly a set of equations obtained by differencing equations for adjacent layers. On the other hand, if we wanted to treat both the external mode and the fastest internal mode ($c^2 = \epsilon gH$) implicitly, then the weighting pattern shown in the table would have to be used.

The consequences of approximations in isolating the external mode for implicit treatment.

We have so far concentrated on approximations, under the assumption that density differences among layers are small. This is not a bad approximation if our models are to represent the atmosphere. As I recall, N. Phillips through similarity considerations determined that $\rho_1/\rho_3 \cong .9$ if the two-layer model is to represent the atmosphere.

However, if the simple weights (16b), (16c), (17b), (17c), in the case of the two-layer model are used to approximate the isolation of the external mode from the internal, what will the nature of the approximation be? The resulting set for the internal mode will include the effects of the external to some degree, for precise isolation can be accomplished only through the weights as given by (13) and (14).

In the context of this discussion we want to know, when the external mode isolated approximately is treated implicitly and the approximately isolated internal mode is treated explicitly, whether what remains of the external mode in the explicit treatment is manifested by (1) a small-amplitude mode with the full speed of the external, or (2) a mode with a reduced speed. It is case (2) which will allow a longer time step than a fully explicit treatment. If case (1) were true, the external mode would have to be isolated precisely, which is rather complicated even for the relatively simple physical systems with which we have been dealing.

To investigate the problem, we return to the two-layer model of the introduction, and let $H_1 = H_3 = H$. To approximately isolate the external mode we add equations (1) and (2) and equations (3) and (4).

$$\frac{\partial}{\partial t}(u_1 + u_3) + g\frac{\partial}{\partial x}\left[(2-\varepsilon)h_1 + 2h_3\right] = 0$$

$$\frac{\partial}{\partial t}(h_1 + h_3) + H\frac{\partial}{\partial x}(u_1 + u_3) = 0$$

These we treat implicitly

$$(u_1 + u_3)_{2t} + g \left[(2-\epsilon)\,\overline{(h_1)_{2x}}^{2t} + 2\,\overline{(h_3)_{2x}}^{2t} \right] = 0$$

$$(h_1)_{2t} + (h_3)_{2t} + H\,\overline{(u_1 + u_3)_{2x}}^{2t}$$

where $\epsilon = 1 - \frac{\rho_1}{\rho_3}$.

Next we approximately isolate the internal by subtracting instead of adding, and treat the result explicitly.

$$(u_1 - u_3)_{2t} + g\epsilon (h_1)_{2x} = 0$$

$$(h_1)_{2t} - (h_3)_{2t} + H (u_1 - u_3)_{2x} = 0$$

Consider

$$h_1 = \overline{h_1} \exp i (qt + rx)$$
$$h_3 = \overline{h_3} \exp i (qt + rx)$$
$$u_1 = U_1 \exp i (qt + rx)$$
$$u_3 = U_3 \exp i (qt + rx)$$

The frequency equation is then

$$0 = (4-\epsilon R^2 + 4R^2) \sin^4 q\Delta t - (4+\epsilon R^2) R^2 \sin^2 q\Delta t + \epsilon R^4$$

where $$R^2 = \frac{2gH \cdot (\Delta t)^2 \sin^2 r\Delta x}{(\Delta x)^2}$$

or

$$\sin^2 q\Delta t = R^2 \frac{4+\epsilon R^2 \pm \sqrt{(4-\epsilon R^2 - 4\epsilon)(4-\epsilon R^2)}}{2(4-\epsilon R^2 + 4R^2)}$$

For stability, a necessary condition is that the radicand be positive. This will be so if either both factors are positive or both factors are negative. If the larger of the two is negative both are, but this condition leads to

$$R^2 \geq \frac{4}{\epsilon}$$

which cannot generally be satisfied, for $\sin^2 r\Delta x$, and therefore R^2, is not limited away from zero. On the other hand, if the smaller of the two factors is positive, both are, which leads to

$$4-\epsilon R^2 \geq 4\epsilon$$

or

$$R^2 \leq 4\left(\frac{1}{\epsilon} - 1\right)$$

This condition along with

$$0 \leq \sin^2 q\Delta t \leq 1$$

are necessary and sufficient for stability.

The last condition may be expressed as

$$-R^2(4+\epsilon R^2) \leq \pm R^2\sqrt{(4-\epsilon R^2 - 4\epsilon)(4-\epsilon R^2)}$$

$$\leq 2(4-\epsilon R^2 + 4R^2) - R^2(4+\epsilon R^2)$$

$$= (4-\epsilon R^2)(2+R^2)$$

These are two independent conditions, one for each sign. The left-hand inequality is satisfied automatically for the upper sign, for a positive number is always equal to or greater than a negative number. The right hand inequality is satisfied for the lower sign for the converse reason. We thus have the following two conditions which must be satisfied for stability.

$$\sqrt{(4-\epsilon R^2 - 4\epsilon)(4-\epsilon R^2)} \leq 4+\epsilon R^2$$

$$\sqrt{(4-\epsilon R^2 - 4\epsilon)(4-\epsilon R^2)} \leq (4-\epsilon R^2)(2+R^2)$$

The first of the last two leads to

$$4-\epsilon R^2 \geq -4R^2$$

and the second leads to

$$-(4-\epsilon R^2)(2)(2+2R^2) \leq 4\epsilon R^4$$

both of which are true under the previous condition, i.e., $4-\epsilon R^2 \geq 4\epsilon$.

The necessary and sufficient condition for stability is thus

$$R^2 \le 4\left(\frac{1}{\epsilon} - 1\right)$$

For the worst case, $\sin^2 r\Delta x = 1$, and

$$\frac{\sqrt{2gH}\cdot\Delta t}{\Delta x} \le 2\sqrt{\frac{1}{\epsilon} - 1}$$

This is to be compared with the condition for the fully explicit system for the precisely isolated external mode,

$$\frac{\sqrt{2gH}\cdot(\Delta t)e}{\Delta x} \le 2\sqrt{\frac{1-\sqrt{1-\epsilon}}{2\epsilon}}$$

and with the condition for the internal mode precisely isolated and treated explicitly

$$\frac{\sqrt{2gH}\cdot(\Delta t)i}{\Delta x} \le 2\sqrt{\frac{1+\sqrt{1-\epsilon}}{2\epsilon}}$$

For example, if $\epsilon = .1$

$$\frac{\sqrt{2gH}\cdot\Delta t}{\Delta x} \le 6$$

$$\frac{\sqrt{2gH}\cdot(\Delta t)e}{\Delta x} \le 1.01$$

$$\frac{\sqrt{2gH}\cdot(\Delta t)i}{\Delta x} \le 6.24$$

We thus see a factor about 6 is gained through a modified semi-implicit treatment, and only about 4% is lost in terms of Δt by making the approximations in isolating the two modes.

Reference

Gerrity, J.P., R.D. McPherson, P.D. Polger, 1971: Requiem for an integration procedure. Office Note 53, National Meteorological Center, National Weather Service, NOAA.