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# COMPARATIVE ANALYSIS OF A NEW INTEGRATION METHOD WITH CERTAIN STANDARD METHODS

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Recently, experiments have been made with a new numerical scheme for integrating the primitive equations. The new method may be expressed by reference to the wave equation

$$\frac{\partial \zeta}{\partial t} = i \omega \zeta \tag{1}$$

by writing

$$\zeta_*^{n+1} = \zeta^{n-1} + 2\Delta t \; i \; \omega \; \zeta_*^n \tag{2a}$$

$$\zeta^n = \alpha \; \zeta_*^n + .5(1-\alpha)(\zeta^{n-1} + \zeta_*^{n+1}) \tag{2b}$$

in which the index, n, fixes the time level and $\alpha$ is a fraction less than unity.

When $\alpha$ is set to unity, the scheme is the well-known "leapfrog" method. When $\alpha$ is set to zero, the method reduces to one studied by Kurihara [1] and called by him the "leapfrog-backward" method. To show this last point, (2b) may be rewritten as ($\alpha = 0$)

$$\zeta^{n+1} = .5(\zeta^n + \zeta^n + 2\Delta t \; i \; \omega \; \zeta_*^{n+1})$$

or

$$\zeta^{n+1} = \zeta^n + \Delta t \; i \; \omega \; \zeta_*^{n+1} \tag{2c}$$

If one defines $b = \omega\Delta t$, following Kurihara, the stability criterion for the leapfrog scheme is

$$b \leq 1$$

and for the leapfrog-backward scheme is

$$b \leq .8$$

Two other schemes have been used in numerical integrations of the primitive equations and analyzed by Kurihara. These are the Euler-backward scheme

$$\zeta_*^{n+1} = \zeta^n + \Delta t \; i \; \omega \; \zeta^n \tag{3a}$$

$$\zeta^{n+1} = \zeta^n + \Delta t \; i \; \omega \; \zeta_*^{n+1} \tag{3b}$$

for which the stability criterion is

$$b < 1.$$

and the "leapfrog-trapezoidal" method

$$\zeta_*^{n+1} = \zeta^{n-1} + 2\Delta t \; i \; \omega \; \zeta^n \tag{4a}$$

$$\zeta^{n+1} = \zeta^n + .5 \; \Delta t\left(i \; \omega \; \zeta^n + i \; \omega \; \zeta_*^{n+1}\right) \tag{4b}$$

One may show that the general scheme (2) provides a solution, $\zeta_\alpha^n$, of the form

$$\zeta_\alpha^n = (1-\alpha)\zeta_{L.B.}^n + \alpha \; \zeta_L^n \tag{5}$$

where $\zeta_{L.B.}^n$ is the result of integration with the leapfrog-backward method, and $\zeta_L^n$ is the result of integration with the leapfrog method.

Now, interest has been expressed in the results to be expected with the method (2) for a variety of values of $\alpha$. It should be noted that (5) does <u>not</u> necessarily imply stability of the new method whenever the criteria for the leapfrog-backward and leapfrog methods are satisfied separately. Therefore, we made calculations to solve the initial value problem,

$$\frac{\partial \zeta}{\partial t} = i \; \omega \; \zeta \tag{6}$$

$$\zeta \text{ at } t = 0 \text{ is } \hat{\zeta} = 1 + 0 \; i, \tag{7}$$

with each of the methods discussed above. The starting procedure for use with method (2) was

$$\zeta_*^1 = \hat{\zeta} + i \; \omega \; \Delta t \; \hat{\zeta} \tag{8a}$$

$$\zeta^0 = \alpha \; \hat{\zeta} + .5(1-\alpha)(\hat{\zeta}+\zeta_*^1) \tag{8b}$$

We defined

$$R = \frac{2\pi}{\omega \Delta t}$$

which implies that the period of the wave is $R$ intervals of time measured in $\Delta t$-units. The amplitude of the solution after 15 steps is tabulated below for various values of $R$ and $\alpha$:

| $\alpha \backslash R$ | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1. | >100. | 1.22 | 1.16 | 1.03 | 1.07 | 1.08 | 1.02 | 1.00 | 1.00 | 1.00 |
| .999 | >100. | 1.21 | 1.15 | 1.03 | 1.07 | 1.08 | 1.02 | 1.00 | 1.00 | 1.00 |
| .990 | >100. | 1.15 | 1.12 | 1.03 | 1.06 | 1.07 | 1.02 | 1.00 | 1.00 | 1.00 |
| .900 | >100. | .97 | .94 | .99 | .97 | 1.00 | 1.00 | .99 | 1.00 | 1.00 |
| .75 | >100. | .56 | .71 | .79 | .84 | .88 | .90 | .92 | .99 | 1.00 |
| .50 | >100. | 2.71 | .32 | .49 | .61 | .69 | .75 | .79 | .97 | .99 |
| .25 | >100. | 41.00 | .11 | .21 | .36 | .48 | .57 | .64 | .94 | .99 |
| 0.0 | >100. | >100. | 4.42 | .11 | .12 | .25 | .36 | .45 | .89 | .97 |
| -.25 | >100. | >100. | 35.44 | 2.74 | .34 | .06 | .12 | .22 | .83 | .96 |
| E.B. | 2.13 | .13 | .13 | .19 | .27 | .35 | .43 | .50 | .89 | .97 |
| L.T. | .23 | .56 | .75 | .86 | .91 | .94 | .96 | .97 | 1.00 | 1.00 |

It will be noted that the leapfrog method yields amplitudes greater than unity even for $R > 2\pi$, the computational stability criterion corresponding to $b \leq 1$. This error is associated with the amplification produced by the "forward," starting scheme. It will be noted that that error is greatly reduced by using $\alpha = .90$. The empirical result for $\alpha = 0$, suggests that the instability with $R = 8,10$ (should be stable by Kurihara's result when $b < .8$, $R \simeq 8$) is also related to the "forward" start utilized with that method (see eqs. 8) and the greater weight attached to the amplified value of $\zeta^1_*$.

Since both the leapfrog-trapezoidal and Euler-backward methods require the computation of two tendencies to advance the calculation, the scheme with $\alpha = .9$ or $.75$ seems to have considerable merit from an efficiency viewpoint.

# REFERENCE

Kurihara, Y., (1965), "On the Use of Implicit and Iterative Methods for Time Integration of the Wave Equation," *Monthly Weather Review*, 93:1, pp 33-46.