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AN ANALYSIS OF THE FREE MODES OF ONE, TWO AND FOUR LAYER MODELS BASED ON SIGMA COORDINATES

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1. Introduction

It is well-known that the computational stability of explicit schemes for numerical integration of the quasi-static equations is dependent upon the satisfaction of a criterion of the form,

$$\omega \Delta t < 1 \tag{1}$$

The parameter, ω, represents the frequency of oscillation of a wave motion admitted by the difference equations. It is seen that (1) requires the use of a smaller time step when a high frequency oscillation is admitted by the system of equations.

In connection with our work on the development of semi-implicit integration methods (1), the need arose for a good estimate of the frequencies admitted by multi-level quasi-static models. In particular, we were led to enquire into possible differences in the modes when alternate formulations of the vertical coordinate were employed.

In this paper, we shall present the results of analyses of the free modes admitted in one, two and four layer models based upon the σ-coordinate systems introduced by Phillips (2), and Shuman and Hovermale (3). The perturbation equations will be developed using an unspecified σ-type coordinate. The specialization to specific coordinates and vertical resolution will then be made, and the analysis indicated. In the concluding section, the results will be summarized and their significance evaluated.

2. The Linear Equations in σ-Coordinates

The equations presented by Shuman and Hovermale (3) pose the physical laws which govern the quasi-static atmosphere. The vertical coordinate, σ, may be chosen as any single-valued, monotonic function of the vertical geometric coordinate. The definitions of σ adopted by Shuman and Hovermale are quite elaborate and were apparently designed to allocate the vertical resolution of their numerical model in a controlled fashion.

In this section, we shall develop a greatly simplified set of equations from those presented in (3). These simplified equations will be specialized by definition of σ and will then be studied in subsequent sections.

At the outset, we assume that the Earth is an infinite, flat plane which is not rotating. One may therefore set the map factor to a constant, unity, and omit the Coriolis terms in Shuman and Hovermale's equations. We then assume that the atmosphere is free from extraneous heat sources, is dry and inviscid. The equations then become

$$\frac{\partial \mathbb{V}}{\partial t} + \mathbb{V}\cdot\nabla\mathbb{V} + \dot{\sigma}\,\frac{\partial \mathbb{V}}{\partial \sigma} + c_p\theta\nabla\pi + \nabla\phi = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial \sigma} + c_p\theta\,\frac{\partial \pi}{\partial \sigma} = 0 \tag{3}$$

$$\frac{\partial}{\partial t}\,\frac{\partial p}{\partial \sigma} + \nabla\cdot\left(\mathbb{V}\,\frac{\partial p}{\partial \sigma}\right) + \frac{\partial}{\partial \sigma}\,\left(\dot{\sigma}\,\frac{\partial p}{\partial \sigma}\right) = 0 \tag{4}$$

$$\frac{\partial \theta}{\partial t} + \mathbb{V}\cdot\nabla\theta + \dot{\sigma}\,\frac{\partial \theta}{\partial \sigma} = 0 \tag{5}$$

with the thermodynamic relationships and definitions,

$$\theta = T\pi^{-1} \tag{6}$$

$$\pi = (p/p_\circ)^{R/c_p} \tag{7}$$

$$p\alpha = RT \tag{8}$$

For our purpose, the thermodynamic parameters used in (2), (3) and (5) are inconvenient. One may show that

$$c_p\theta\nabla\pi = \alpha\nabla p \tag{9}$$

$$c_p\theta\,\frac{\partial \pi}{\partial \sigma} = \alpha\frac{\partial p}{\partial \sigma} \tag{10}$$

and that the thermodynamic equation (5) may be re-expressed

$$c_p\,\left(\frac{\partial T}{\partial t} + \mathbb{V}\cdot\nabla T + \dot{\sigma}\,\frac{\partial T}{\partial \sigma}\right) = \alpha\,\left(\frac{\partial p}{\partial t} + \mathbb{V}\cdot\nabla p + \dot{\sigma}\,\frac{\partial p}{\partial \sigma}\right) \tag{11}$$

When (9) and (10) are introduced into (2) and (3), the latter become,

$$\frac{\partial \mathbb{V}}{\partial t} + \mathbb{V}\cdot\nabla\mathbb{V} + \dot{\sigma}\,\frac{\partial \mathbb{V}}{\partial \sigma} + \alpha\nabla p + \nabla\phi = 0 \tag{12}$$

and

$$\frac{\partial \phi}{\partial \sigma} + \alpha\,\frac{\partial p}{\partial \sigma} = 0 \tag{13}$$

We proceed now to define a basic state which is barotropic and involves no wind. Letting the basic state variables be denoted by a superscript bar, we find

$$
\begin{aligned}
\overline{V} &= 0 \\
\overline{\dot{\sigma}} &= 0 \\
\overline{p}\overline{\alpha} &= R\overline{T} \\
\nabla\overline{\alpha} = \nabla\overline{T} = \nabla\overline{p} &= \nabla\overline{\phi} = 0 \\
\frac{\partial\overline{\phi}}{\partial\sigma} &= -\overline{\alpha}\,\frac{\partial\overline{p}}{\partial\sigma}
\end{aligned}
\tag{14}
$$

will be sufficient to insure that the basic state is steady.

We now define a perturbation field in each dependent variable and assume that it does not vary in the y direction. For ease of notation, the customary prime on the perturbations will be omitted; all parameters without overbars will be understood to be perturbations. So, replacing the variables with the sum of a basic state and a perturbation, the equations become

$$\frac{\partial u}{\partial t} + \overline{\alpha}\,\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial x} \tag{15}$$

$$\frac{\partial \phi}{\partial \sigma} + \overline{\alpha}\,\frac{\partial p}{\partial \sigma} + \alpha\,\frac{\partial \overline{p}}{\partial \sigma} = 0 \tag{16}$$

$$\frac{\partial}{\partial t}\,\frac{\partial p}{\partial \sigma} + \dot{\sigma}\,\frac{\partial^2 \overline{p}}{\partial \sigma^2} + \frac{\partial \overline{p}}{\partial \sigma}\left(\frac{\partial u}{\partial x} + \frac{\partial \dot{\sigma}}{\partial \sigma}\right) = 0 \tag{17}$$

$$c_p\,\frac{\partial T}{\partial t} - \overline{\alpha}\,\frac{\partial p}{\partial t} + c_p\,\dot{\sigma}\left(\frac{\partial \overline{T}}{\partial \sigma} - \frac{\overline{\alpha}}{c_p}\,\frac{\partial \overline{p}}{\partial \sigma}\right) = 0 \tag{18}$$

$$\overline{p}\alpha + p\overline{\alpha} = RT \tag{19}$$

This set of linear equations in six dependent variables is not closed. It remains for one to define the vertical coordinate, at least implicitly, in terms of z. A subtle point in the linearization involves the question of the existence of a perturbation in the σ-variable itself. We have followed the view that no such perturbation is admissible. Once a coordinate has been selected, it is immutable from the observer's viewpoint. Thus, if pressure is selected as the vertical coordinate, one cannot "observe" a pressure perturbation. On the contrary, one observes perturbations in geopotential. In the "inertial" frame, in which a geometric altitude coordinate is used, the observation in the pressure coordinate is seen to be distorted. An analogy to the Coriolis acceleration is obvious.

With this point stressed, we may proceed to specialize the linear equations to various σ-systems, and to investigate the free modes admitted by the equations.

In the case of Phillips definition of σ,

$$\sigma = p/p^* = \overline{p}/\overline{p^*} \quad 0 \leq p \leq p^* \tag{20}$$

with p^* the pressure on the ground, the system of equations becomes,

$$\frac{\partial u}{\partial t} + \overline{\alpha}\,\sigma\,\frac{\partial p^*}{\partial x} + \frac{\partial \phi}{\partial x} = 0 \tag{21}$$

$$\frac{\partial \phi}{\partial \sigma} + \overline{\alpha}\,p^* + \alpha\,\overline{p^*} = 0 \tag{22}$$

$$\frac{\partial}{\partial t}\,p^* + \overline{p^*}\left(\frac{\partial u}{\partial x} + \frac{\partial \dot{\sigma}}{\partial \sigma}\right) = 0 \tag{23}$$

$$c_p\,\frac{\partial T}{\partial t} - \overline{\alpha}\,\sigma\,\frac{\partial p^*}{\partial t} + c_p\,\dot{\sigma}\,\overline{\Gamma} = 0 \tag{24}$$

$$\sigma\,\overline{p^*}\,\alpha + \sigma\,p^*\,\overline{\alpha} = RT \tag{25}$$

$$\overline{\Gamma} \equiv \frac{\partial \overline{T}}{\partial \sigma} - \frac{\overline{\alpha}}{c_p}\,\overline{p^*} \tag{26}$$

Shuman and Hovermale use a number of definitions for σ within different portions of the atmosphere. We shall here consider only a two-domain case; thus, σ is defined by

$$\sigma_s = p/p^T = \overline{p}/\overline{p^T}\ , \ 0 \leq p \leq p^T \tag{27}$$

and

$$\begin{aligned}\sigma_T &= (p-p^T)/(p^*-p^T)\\ &= (\overline{p}-\overline{p^T})/(\overline{p^*}-\overline{p^T}),\ p^T \leq p \leq p^* \end{aligned}\tag{28}$$

One may imagine (27) to prescribe the vertical coordinate above the tropopause and (28) to define σ within the troposphere. It is clear that the equations will have the same form as the Phillips system above the tropopause, provided that p^T replaces p^* and σ_T replaces σ. The system generated by use of (28) is,

$$\frac{\partial u}{\partial t} + \bar{\alpha}\,\sigma_T \frac{\partial}{\partial x}\left(p^* - p^T\right) + \bar{\alpha}\,\frac{\partial}{\partial x} p^T + \frac{\partial \phi}{\partial x} = 0 \tag{29}$$

$$\frac{\partial \phi}{\partial \sigma} + \bar{\alpha}\left(p^* - p^T\right) + \alpha\left(\overline{p^*} - \overline{p^T}\right) = 0 \tag{30}$$

$$\frac{\partial}{\partial t}\left(p^* - p^T\right) + \left(\overline{p^*} - \overline{p^T}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial \dot{\sigma}}{\partial \sigma}\right) = 0 \tag{31}$$

$$c_p \frac{\partial T}{\partial t} - \sigma_T\,\bar{\alpha}\,\frac{\partial}{\partial t}\left(p^* - p^T\right) - \bar{\alpha}\,\frac{\partial}{\partial t} p^T + c_p\,\dot{\sigma}\,\bar{\Gamma}_T = 0 \tag{32}$$

$$\sigma_T\left(\overline{p^*} - \overline{p^T}\right)\alpha + \overline{p^T}\,\alpha + \sigma_T\left(p^* - p^T\right)\bar{\alpha} + p^T\,\bar{\alpha} = RT \tag{33}$$

$$\bar{\Gamma}_T = \frac{\partial \bar{T}}{\partial \sigma} - \frac{\bar{\alpha}}{c_p}\left(\overline{p^*} - \overline{p^T}\right) \tag{34}$$

Finally, we shall be seeking the free modes admitted by the system of linear equations. In other words, we shall try to find the values of c for which solutions of the form

$$q(x,y,\sigma,t) = q(\sigma)\,e^{ik(x+ct)} \tag{35}$$

will satisfy the equations. Rather than develop differential equations for $q(\sigma)$, the equations will be discretized by use of finite difference approximations of the vertical dependence. In this way, one develops a set of simultaneous, linear, algebraic equations with c as a parameter. The requirement that (35) be non-trivial imposes the condition that the determinant of the system of equations vanish. The determinant will yield a polymonial in c with the basic state variables entering as coefficients. The roots of the polymonial (the allowable modes) will be determined by specifying the basic state.

3. The One-Layer Phillips System

The equations (20) through (26) may be expressed upon introduction of (35) as,

$$\sigma = p/p^* = \bar{p}/\overline{p^*} \qquad 0 \leq p \leq p^* \tag{36}$$

$$c\,u + \bar{\alpha}\,\sigma\,p^* + \phi = 0 \tag{37}$$

$$\frac{\partial \phi}{\partial \sigma} + \bar{\alpha}\,p^* + \alpha\,\overline{p^*} = 0 \tag{38}$$

$$c\ p^* + \overline{p^*}\ \left\{u + \frac{\partial}{\partial\sigma}\ \omega\right\} = 0 \tag{39}$$

$$c_p\ c\ T - c\ \overline{\alpha}\ \sigma\ p^* + c_p\ \omega\ \overline{\Gamma} = 0 \tag{40}$$

$$\sigma\ \overline{p^*}\ \alpha + \sigma\ p^*\ \overline{\alpha} = RT \tag{41}$$

$$\overline{\Gamma} \equiv \frac{\partial \overline{T}}{\partial\sigma} - \frac{\overline{\alpha}}{c_p}\ \overline{p^*} \tag{42}$$

in which $\omega \equiv (i\ k)^{-1}\ \dot{\sigma}$ (43)

The discretization of this system for a one layer model is indicated in the diagram below,

$\sigma = 0$	$\dot{\sigma} = 0$	$p = 0$	$\phi = \phi_1$
$\sigma = \frac{1}{2}$	$u_1,\ \alpha_1,\ T_1$	$p_1 = \frac{1}{2}\ p^*$	
$\sigma = 1$	$\dot{\sigma} = 0$	$p = p^*$	$\phi = 0$

So the equations become,

$$c\ u_1 + .5\ \overline{\alpha}_1\ p^* + .5\ \phi_1 = 0 \tag{44}$$

$$-\ \phi_1 + \overline{\alpha}_1\ p^* + \alpha_1\ \overline{p^*} = 0 \tag{45}$$

$$c\ p^* + \overline{p^*}\ (u_1 + 0) = 0 \tag{46}$$

$$c_p\ c\ T_1 - c\ \overline{\alpha}_1 .5\ p^* + 0 = 0 \tag{47}$$

$$.5\ \overline{p^*}\ \alpha_1 + .5\ p^*\ \overline{\alpha}_1 = RT_1 \tag{48}$$

We may now eliminate T_1 and α_1 by use of (45), (47) and (48) to get

$$+\ \phi_1 = \overline{\alpha}_1\ \kappa\ p^* \tag{49}$$

with $\kappa = {}^R/c_p$.

Then ϕ_1 and u_1 may be eliminated between (44), (46) and (49) to get

$$\left\{c^2 - (\kappa + 1)\ R\overline{T}_1\right\} p^* = 0 \tag{50}$$

in which we have used

$$.5\ \overline{\alpha}_1\ \overline{p^*} = R\overline{T}_1 \tag{51}$$

Thus, the free modes for the one-layer Phillips model are

$$c = \pm \left((\kappa + 1)\ R\overline{T}_1\right)^{\frac{1}{2}} \tag{52}$$

If $\overline{T}_1 = 250°K$, one finds

$$c \simeq \pm\ 302 \text{ m sec}^{-1}\ .$$

4. <u>One-Layer Shuman System</u>

It is clear that the Shuman system for defining σ is not appropriate for use in a one-layer model. It is of interest, however, to examine the result of bounding the atmosphere at a pressure other than zero. We shall, therefore, consider the system of equations which result from the assumptions

$$\left.\begin{aligned} \overline{p^T} &= \text{constant} \\ p^T &= 0 \end{aligned}\right. \tag{53}$$

The diagram below indicates the vertical discretization used in eqs. (28) through (34), using (53) and (35),

$\sigma_T = 0$	$\dot{\sigma} = 0 \quad \phi = \phi_1$	$\overline{p} = \overline{p^T}\ ,\quad p^T = 0 = p$
$\sigma_T = .5$	$u_1,\ \alpha_1,\ T_1$	$\overline{p}_1 = .5(\overline{p^*} + \overline{p^T}),\ p_1 = .5\ p^*$
$\sigma_T = 1$	$\dot{\sigma} = 0,\quad \phi = 0$	$\overline{p} = \overline{p^*};\ p = p^*$

The equations reduce to,

$$c\ u_1 + .5\ \overline{\alpha}_1\ p^* + .5\ \phi_1 = 0 \tag{54}$$

$$-\ \phi_1 + \overline{\alpha}_1\ p^* + \alpha_1\ (\overline{p^*} - \overline{p^T}) = 0 \tag{55}$$

$$c\ p^* + (\overline{p^*} - \overline{p^T})[u_1 + 0] = 0 \tag{56}$$

$$c_p\ c\ T_1 - .5\ \overline{\alpha}_1\ c\ p^* = 0 \tag{57}$$

$$.5(\overline{p^*} - \overline{p^T})\ \alpha_1 + \overline{p^T}\ \alpha_1 + .5\ p^*\ \overline{\alpha}_1 = R T_1 \tag{58}$$

One may eliminate α_1 and T_1 among equations (55), (57) and (58) to obtain

$$\phi_1 = \left(1 - (1 - \kappa)\ \varepsilon\right) \overline{\alpha}_1\ p^* \tag{59}$$

in which

$$\varepsilon \equiv \left(\overline{p^*} - \overline{p^T}\right)/\left(\overline{p^*} + \overline{p^T}\right) \tag{60}$$

Then u_1 and ϕ_1 may be eliminated between (54), (56) and (59), yielding:

$$\left\{c^2 - (\overline{p^*} - \overline{p^T})\ \overline{\alpha}_1\ [1 - .5\ \varepsilon(1 - \kappa)]\right\} p^* = 0$$

or

$$\left\{c^2 - 2\ \varepsilon\ R\overline{T}_1\ [1 - .5(1 - \kappa)\ \varepsilon]\right\} p^* = 0 \tag{61}$$

The roots of (61) are,

$$c = \pm \left(\varepsilon\ R\overline{T}_1\ [2 - \varepsilon + \varepsilon\ \kappa]\right)^{1/2} \tag{62}$$

If $\overline{p^T} = 0$, then $\varepsilon = 1$ and equation (62) reduces to the result obtained in the Phillips one-layer model. Some interesting choices of ε yield the following results with $\kappa = .287$,

$$c_{.3} = \pm \left(R\overline{T}_1\ .587\right)^{1/2}, \quad \varepsilon = {}^1/_3$$

$$c_{.6} = \pm \left(R\overline{T}_1\ .952\right)^{1/2}, \quad \varepsilon = {}^2/_3$$

$$c_1 = \pm \left(R\overline{T}_1\ 1.287\right)^{1/2}, \quad \varepsilon = 1$$

5. Phillips System Two-Layer Model

The diagram below indicates the vertical discretization to be used:

$\sigma = 0$ ——— $\dot{\sigma} = 0$ ——— ϕ_2 $\quad p = 0$

- - - u_2, α_2, T_2 - - -

$\sigma = .5$ ——— $\dot{\sigma}$ ——— ϕ_1 $\quad p = p^*/2$

- - - u_1, α_1, T_1 - - -

$\sigma = 1$ ——— $\dot{\sigma} = 0$ ——— $\phi = 0$ $\quad p = p^*$

$$\overline{\Gamma} = 2\left(\overline{T}_1 - \overline{T}_2\right) - \frac{\overline{\alpha}_1 + \overline{\alpha}_2}{2\, c_p}\, \overline{p^*} \tag{63}$$

We find it convenient to define,

$$r = p^*/\overline{p^*} \tag{64}$$

and

$$\omega = \dot{\sigma}/ik \tag{65}$$

The system of equations may then be written,

$$c\, u_1 + .5\, \phi_1 + R\overline{T}_1\, r = 0 \tag{66}$$

$$c\, u_2 + .5\,(\phi_2 + \phi_1) + R\overline{T}_2 r = 0 \tag{67}$$

$$2\, c\, r + (u_1 + u_2) = 0 \tag{68}$$

$$(u_1 - u_2) - 4\omega = 0 \tag{69}$$

$$.75\, \overline{p^*}\, \alpha_1 = 1.5\, \phi_1 - R\overline{T}_1 r \tag{70}$$

$$.25\, \overline{p^*}\, \alpha_2 = .5\,(\phi_2 - \phi_1) - R\overline{T}_2 r \tag{71}$$

$$cRT_1 = c\, \kappa\, R\overline{T}_1 r - .5\, R\, \overline{\Gamma}\, \omega \tag{72}$$

$$cRT_2 = c\, \kappa\, R\overline{T}_2 r - .5\, R\, \overline{\Gamma}\, \omega \tag{73}$$

$$RT_1 = .75\, \overline{p^*}\, \alpha_1 + R\overline{T}_1\, r \tag{74}$$

$$RT_2 = .25\ \overline{p^*}\ \alpha_2 + R\overline{T}_2 r \tag{75}$$

Elimination of α_1, α_2 and T_1, T_2 among equations (70) through (75), yields

$$1.5\ c\ \phi_1 = c\ \kappa\ R\overline{T}_1 r - R\overline{\Gamma}\ .5\ \omega \tag{76}$$

$$.5\ c\ [\phi_2 - \phi_1] = c\ \kappa\ R\overline{T}_2 r - R\overline{\Gamma}\ .5\ \omega \tag{77}$$

We have assumed that $c = 0$ is not a root.

One may also eliminate u_1 and u_2 among equations (66) through (69), to get

$$[2\ c^2 - (R\overline{T}_1 + R\overline{T}_2)]\ r - \phi_1 - .5\ \phi_2 = 0 \tag{78}$$

$$[R\overline{T}_1 - R\overline{T}_2]\ r + 4\ c\ \omega - .5\ \phi_2 = 0 \tag{79}$$

If one next works out the determinant of the coefficients of equations (76) through (79) and sets it equal to zero, one finds that c must satisfy the equation,

$$6\ c^4 - [3(\kappa + 1)(R\overline{T}_1 + R\overline{T}_2) - R\overline{\Gamma}]\ c^2$$
$$- R\overline{\Gamma}\ [\tfrac{1}{4}(1 + \kappa)(R\overline{T}_1 - R\overline{T}_2) + R\overline{T}_1] = 0 \tag{80}$$

If $\overline{\Gamma} = 0$, two of the roots of (80) are found to be zero, the other two are similar to those for the one-layer model but with

$$\overline{T} = (\overline{T}_1 + \overline{T}_2)/2$$

For an isothermal basic state,

$$R\overline{\Gamma} = -\frac{\kappa}{2}\left(\frac{R\overline{T}_1}{.75} + \frac{R\overline{T}_2}{.25}\right)$$

$$= -\frac{8}{3}\ \kappa\ R\overline{T} \tag{81}$$

and

$$\overline{T}_1 = \overline{T}_2 = \overline{T} \tag{82}$$

Substitution of (81) and (82) into (80), gives the isothermal relation,

$$c^4 - [(\kappa + 1) + \frac{8\kappa}{18}] R\overline{T} c^2 + \frac{8\kappa}{18} (R\overline{T})^2 = 0 \tag{83}$$

If one uses $\kappa = 2/7$, (83) becomes

$$c^4 - \left(\frac{178}{126}\right) R\overline{T} c^2 + \left(\frac{16}{126}\right) (R\overline{T})^2 = 0 \tag{84}$$

with the roots

$$\begin{aligned} c_1 &= \pm \left(\frac{331.68}{252} R\overline{T} \right)^{\frac{1}{2}} = \pm [1.31 R\overline{T}]^{\frac{1}{2}} \\ c_2 &= \pm \left(\frac{24.32}{252} R\overline{T} \right)^{\frac{1}{2}} = \pm [.09 R\overline{T}]^{\frac{1}{2}} \end{aligned} \tag{85}$$

The larger of these roots is slightly greater than that obtained in the one-layer model. The smaller root is the internal mode. For $\overline{T} = 250°K$ one gets

$$c_1 = \pm 306. \quad m\ sec^{-1}$$

$$c_2 = \pm 81. \quad m\ sec^{-1}$$

6. The Shuman Two-Layer System

The diagram below indicates the vertical discretization appropriate to the two-layer model using Shuman's system,

	$\sigma_s = 0$	$\dot{\sigma} = 0 \quad \phi \quad \phi_2$	$p = 0$
	$\sigma_s = .5$	u_2, α_2, T_2	$p = .5\ p^T$
$\sigma_s = 1$	$\sigma_T = 0$	$\dot{\sigma} = 0 \quad \phi = \phi_1$	$p = p^T$
	$\sigma_T = .5$	u_1, α_1, T_1	$p = .5\ (p^* + pT)$
	$\sigma_T = 1$	$\dot{\sigma} = 0 \quad \phi = 0$	$p = p^*$

The system of equations may be written

$$c\, u_1 + .5(\phi_1) + \overline{\alpha}_1\ .5(p^* + p^T) = 0 \tag{86}$$

$$c\, u_2 + .5(\phi_1 + \phi_2) + \overline{\alpha}_2\ .5\ p^T = 0 \tag{87}$$

$$c\ (p^* - p^T) + (\overline{p^*} + \overline{p^T})u_1 = 0 \tag{88}$$

$$c\ p^T + \overline{p^T}\ \ u_2 = 0 \tag{89}$$

$$-\ \phi_1 + \overline{\alpha}_1\ (p^* - p^T) + \alpha_1\ (\overline{p^*} - \overline{p^T}) = 0 \tag{90}$$

$$-\ (\phi_2 - \phi_1) + \overline{\alpha}_2\ p^T + \alpha_2\ \overline{p^T} = 0 \tag{91}$$

$$c_p\ T_1 - \alpha_1\ .5\ (p^* - p^T) = 0 \tag{92}$$

$$c_p\ T_2 - \overline{\alpha}_2\ .5\ p^T = 0 \tag{93}$$

$$.5(\overline{p^*} + \overline{p^T})\ \alpha_1 + .5(p^* + p^T)\ \overline{\alpha}_1 = RT_1 \tag{94}$$

$$.5\ \overline{p^T}\ \alpha_2 + .5\ p^T\ \overline{\alpha}_2 = RT_2 \tag{95}$$

One may eliminate α_1, α_2, T_1 and T_2 among equations (90) through (95),

$$+\ \phi_1 = \overline{\alpha}_1\ (p^* - p^T) + \overline{\alpha}_1\ \varepsilon\ (\kappa - 1)(p^* + p^T) \tag{96}$$

and $$\phi_2 = \phi_1 + \kappa\ \overline{\alpha}_2\ p^T \tag{97}$$

with

$$\varepsilon \equiv (\overline{p^*} - \overline{p^T})/(\overline{p^*} + \overline{p^T}) \tag{98}$$

One may also eliminate u_1 and u_2 among equations (86) through (89) to obtain,

$$c^2\ (p^* - p^T) - (\overline{p^*} - \overline{p^T})\ \overline{\alpha}_1\ .5(p^* + p^T) - .5(\overline{p^*} - \overline{p^T})\ \phi_1 = 0 \tag{99}$$

and

$$(c^2 - .5\ \overline{\alpha}_2\ \overline{p^T})\ p^T - .5\ \overline{p^T}\ \phi_1 - .5\ \overline{p^T}\ \phi_2 = 0 \tag{100}$$

Finally, ϕ_1 and ϕ_2 may be eliminated by use of (96) and (97) in (99) and (100), to get

$$\left(c^2 - \varepsilon\ R\overline{T}_1\ [2 + \varepsilon(\kappa - 1)]\right)p^* - [c^2 + \varepsilon^2 R\overline{T}_1(\kappa - 1)]p^T = 0 \tag{101}$$

and

$$\{c^2 - R\overline{T}_2 [1 + \kappa - 2\ r \{1 - \varepsilon(\kappa - 1)\}]\}p^T$$
$$- 2\ r\ R\overline{T}_2 [1 + \varepsilon(\kappa - 1)]p^* = 0 \qquad (102)$$

in which

$$r \equiv \overline{\alpha}_1 / \overline{\alpha}_2 = (\overline{T}_1/\overline{T}_2)\left(\overline{p^T}/(\overline{p^*} + \overline{p^T})\right) \qquad (103)$$

The determinant of the coefficients of equations (101) and (102) must vanish, giving rise therefore to a fourth degree polynomial in c.

If the basic state is specified to be isothermal ($\overline{T}$ = 250°K) and the pressures are chosen so that

$$\overline{p^T} = \overline{p^*} - \overline{p^T},$$

then one obtains

$$\varepsilon = r = 1/3 .$$

The roots of the c polynomial are then

$$c = \pm\ 308 \text{ m sec}^{-1}$$

$$c = \pm\ 130 \text{ m sec}^{-1}$$

It will be noted that the larger of these roots are sensibly the same as those obtained for the Phillips two-layer system. The smaller roots of the Shuman system are considerably larger than those internal mode values obtained with the Phillips system.

7. The Four-Layer Phillips System

The vertical discretization used in this case is shown in the diagram below

$\sigma = 0$	$\dot{\sigma} = 0$		ϕ_4, $p = 0$
	u_4, T_4, α_4		
$\sigma = .25$	$\dot{\sigma}_3$	$\bar{\Gamma}_{34}$	ϕ_3, $p = .25\ p^*$
	u_3, T_3, α_3		
$\sigma = .50$	$\dot{\sigma}_2$	$\bar{\Gamma}_{23}$	ϕ_2, $p = .50\ p^*$
	u_2, T_2, α_2		
$\sigma = .75$	$\dot{\sigma}_1$	$\bar{\Gamma}_{12}$	ϕ_3, $p = .75\ p^*$
	u_1, T_1, α_1		
$\sigma = 1.00$	$\dot{\sigma} = 0$		$\phi = 0$, $p = p^*$

The system of equations may be written,

$$
\begin{aligned}
&c\,u_1 + .5\,\phi_1 + \bar{\alpha}_1\,p_1 = 0\\
&c\,u_2 + .5\,\phi_1 + .5\,\phi_2 + \bar{\alpha}_2\,p_2 = 0\\
&c\,u_3 + .5\,\phi_2 + .5\,\phi_3 + \bar{\alpha}_3\,p_3 = 0\\
&c\,u_4 + .5\,\phi_3 + .5\,\phi_4 + \bar{\alpha}_4\,p_4 = 0
\end{aligned}
\qquad (104)
$$

$$- 4\,\phi_1 + \overline{\alpha}_1\,p^* + \alpha_1\,\overline{p^*} = 0$$

$$4(\phi_1 - \phi_2) + \overline{\alpha}_2\,p^* + \alpha_2\,\overline{p^*} = 0$$

$$4(\phi_2 - \phi_3) + \overline{\alpha}_3\,p^* + \alpha_3\,\overline{p^*} = 0 \qquad (105)$$

$$4(\phi_3 - \phi_4) + \overline{\alpha}_4\,p^* + \alpha_4\,\overline{p^*} = 0$$

$$c\,p^* + \overline{p^*}\,[u_1 - 4(i\,k)^{-1}\,\dot{\sigma}_1] = 0$$

$$c\,p^* + \overline{p^*}\,[u_2 + 4(i\,k)^{-1}\,(\dot{\sigma}_1 - \dot{\sigma}_2)] = 0$$

$$c\,p^* + \overline{p^*}\,[u_3 + 4(i\,k)^{-1}\,(\dot{\sigma}_2 - \dot{\sigma}_3)] = 0 \qquad (106)$$

$$c\,p^* + \overline{p^*}\,[u_4 + 4(i\,k)^{-1}\,\dot{\sigma}_3] = 0$$

$$c\,c_p\,T_1 - c\,\overline{\alpha}_1\,p_1 + c_p(i\,k)^{-1}\,[.5\,\overline{\Gamma}_{12}\,\dot{\sigma}_1] = 0$$

$$c\,c_p\,T_2 - c\,\overline{\alpha}_2\,p_2 + c_p(i\,k)^{-1}\,[.5\,\overline{\Gamma}_{12}\,\dot{\sigma}_1 + .5\,\overline{\Gamma}_{23}\,\dot{\sigma}_2] = 0$$

$$c\,c_p\,T_3 - c\,\overline{\alpha}_3\,p_3 + c_p(i\,k)^{-1}\,[.5\,\overline{\Gamma}_{23}\,\dot{\sigma}_2 + .5\,\overline{\Gamma}_{34}\,\dot{\sigma}_3] = 0 \qquad (107)$$

$$c\,c_p\,T_4 - c\,\overline{\alpha}_4\,p_4 + c_p(i\,k)^{-1}\,[.5\,\overline{\Gamma}_{34}\,\dot{\sigma}_3] = 0$$

$$\overline{p}_1\,\alpha_1 + p_1\,\overline{\alpha}_1 = RT_1$$

$$\overline{p}_2\,\alpha_2 + p_2\,\overline{\alpha}_2 = RT_2$$

$$\overline{p}_3\,\alpha_3 + p_3\,\overline{\alpha}_3 = RT_3 \qquad (108)$$

$$\overline{p}_4\,\alpha_4 + p_4\,\overline{\alpha}_4 = RT_4$$

$$p_1 = 7\,p^*/8$$

$$p_2 = 5\,p^*/8$$

$$p_3 = 3\,p^*/8 \qquad (109)$$

$$p_4 = 1\,p^*/8$$

$$\begin{aligned}
\overline{\Gamma}_{12} &= 4\ (\overline{T}_1 - \overline{T}_2) - (\overline{\alpha}_2 + \overline{\alpha}_1)\ \overline{p^*}/(2\ c_p) \\
\overline{\Gamma}_{23} &= 4\ (\overline{T}_2 - \overline{T}_3) - (\overline{\alpha}_3 + \overline{\alpha}_2)\ \overline{p^*}/(2\ c_p) \qquad (110) \\
\overline{\Gamma}_{34} &= 4\ (\overline{T}_3 - \overline{T}_4) - (\overline{\alpha}_4 + \overline{\alpha}_3)\ \overline{p^*}/(2\ c_p)
\end{aligned}$$

The set of equations just enumerated may be reduced to a set of four, homogeneous equations in the four geopotentials. For a unique, non-trivial solution to exist, we are led to require the satisfaction of a "frequency equation" in c. The zeros of the frequency equation yield the free mode phase speeds, c. Eight values of c were calculated numerically for an isothermal basic state with a mean temperature of 250°K.

The results were,

$$\begin{aligned}
c_1 &= \pm\ 308 \text{ m sec}^{-1} \\
c_2 &= \pm\ 110 \text{ m sec}^{-1} \\
c_3 &= \pm\ 43 \text{ m sec}^{-1} \\
c_4 &= \pm\ 24 \text{ m sec}^{-1}
\end{aligned}$$

It will be noted that the fastest phase speed is essentially the same as that calculated using the one- and two-layer models based on Phillips σ-system. The second mode, c_2, is larger than that obtained with the two-layer model. This difference is attributable to the variation in static stability resolution.

8. The Shuman System Four-Layer Model

The vertical discretization used with the Shuman four-layer model is indicated in the diagram below

ϕ_5	$\dot{\sigma} = 0$	$\sigma_s = 0 \quad p = 0$
	$u_4,\ T_4,\ \alpha_4,\ p_4$	
ϕ_4	$\dot{\sigma}_2$	$\sigma_s = .5 \quad p = .5\, p^T$
	$u_3,\ T_3,\ \alpha_3,\ p_3$	
ϕ_3	$\dot{\sigma} = 0$	$\sigma_s = 1.,\ \sigma_T = 0,\ p = p^T$
	$u_2,\ T_2,\ \alpha_2,\ p_2$	
ϕ_2	$\dot{\sigma}_1$	$\sigma_T = .5,\ p = .5(p^* + p^T)$
	$u_1,\ \alpha_1,\ T_1,\ p_1$	
$\phi = 0$	$\dot{\sigma} = 0$	$\sigma_T = 1.,\ p = p^*$

The system of equations may be written,

$$
\begin{aligned}
c\, u_1 + .5\, \phi_2 \qquad\qquad + \overline{\alpha}_1\, p_1 &= 0 \\
c\, u_2 + .5\, (\phi_2 + \phi_3) + \overline{\alpha}_2\, p_2 &= 0 \\
c\, u_3 + .5\, (\phi_3 + \phi_4) + \overline{\alpha}_3\, p_3 &= 0 \\
c\, u_4 + .5\, (\phi_4 + \phi_5) + \overline{\alpha}_4\, p_4 &= 0
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
c\, (p^* - p^T) + \overline{(p^* - p^T)}\,[u_1 - 2(i\, k)^{-1}\, \dot{\sigma}_1] &= 0 \\
c\, (p^* - p^T) + \overline{(p^* - p^T)}\,[u_2 + 2(i\, k)^{-1}\, \dot{\sigma}_1] &= 0 \\
c\, p^T + \overline{p^T}\, [u_3 - 2(i\, k)^{-1}\, \dot{\sigma}_2] &= 0 \\
c\, p^T + \overline{p^T}\, [u_4 + 2(i\, k)^{-1}\, \dot{\sigma}_2] &= 0
\end{aligned}
\tag{112}
$$

$$-2\phi_2 + \overline{\alpha}_1 (p^* - p^T) + \alpha_1 (\overline{p^* - p^T}) = 0$$
$$2(\phi_2 - \phi_3) + \overline{\alpha}_2 (p^* - p^T) + \alpha_2 (\overline{p^* - p^T}) = 0 \tag{113}$$
$$2(\phi_3 - \phi_4) + \overline{\alpha}_3 p^T + \alpha_3 \overline{p^T} = 0$$
$$2(\phi_4 - \phi_5) + \overline{\alpha}_4 p^T + \alpha_4 \overline{p^T} = 0$$

$$c_p c T_1 - c \overline{\alpha}_1 p_1 + .5(i k)^{-1} c_p \overline{\Gamma^T} \dot{\sigma}_1 = 0$$
$$c_p c T_2 - c \overline{\alpha}_2 p_2 + .5(i k)^{-1} c_p \overline{\Gamma^T} \dot{\sigma}_1 = 0 \tag{114}$$
$$c_p c T_3 - c \overline{\alpha}_3 p_3 + .5(i k)^{-1} c_p \overline{\Gamma^s} \dot{\sigma}_2 = 0$$
$$c_p c T_4 - c \overline{\alpha}_4 p_4 + .5(i k)^{-1} c_p \overline{\Gamma^s} \dot{\sigma}_2 = 0$$

$$\overline{p}_1 \alpha_1 + p_1 \overline{\alpha}_1 = RT_1$$
$$\overline{p}_2 \alpha_2 + p_2 \overline{\alpha}_2 = RT_2 \tag{115}$$
$$\overline{p}_3 \alpha_3 + p_3 \overline{\alpha}_3 = RT_3$$
$$\overline{p}_4 \alpha_4 + p_4 \overline{\alpha}_4 = RT_4$$

$$p_1 = .25\, p^T + .75\, p^* , \quad \overline{p}_1 = .25\, \overline{p^T} + .75\, \overline{p^*}$$
$$p_2 = .75\, p^T + .25\, p^* , \quad \overline{p}_2 = .75\, \overline{p^T} + .25\, \overline{p^*} \tag{116}$$
$$p_3 = .75\, p^T , \quad \overline{p}_3 = .75\, \overline{p^T}$$
$$p_4 = .25\, p^T , \quad \overline{p}_4 = .25\, \overline{p^T}$$

This set of simultaneous equations is closed. It should be noted that the static stability of the basic state has only two distinct values which we have denoted

$$\overline{\Gamma^T} = [2(\overline{T}_1 - \overline{T}_2) - (2 c_p)^{-1} (\overline{p^* - p^T})(\overline{\alpha}_1 + \overline{\alpha}_2)]$$
$$\overline{\Gamma^s} = [2(\overline{T}_3 - \overline{T}_4) - (2 c_p)^{-1} (\overline{p^T}) (\overline{\alpha}_3 + \overline{\alpha}_4)] \tag{117}$$

For a non-trivial solution to exist for the system, one requires the determinant of the matrix of coefficients to vanish.

If we set D to stand for the determinant, we have

$$D(c;\overline{q}_i) = 0 \tag{118}$$

in which $\overline{q}_i$ stands for the basic state parameters. By fixing $\overline{q}_i$, D becomes a polynomial in c. Since only neutral waves are expected, all the roots of D will be real valued. To determine the roots, we simply evaluate D as c varies over the range, 0 to c_{max}. The choice of c_{max} is made on physically realistic bases.

The analysis was made more tractable by simplifying the basic set of equations through the elimination of all the variables except the four geopotentials and the two pressures, p^* and p^T. In order to carry out the elimination, it was necessary to assume that $c \neq 0$.

The simplified set of equations may be expressed as,

$$D\,v = 0$$

with

$$D = \begin{bmatrix} 0 & -\overline{p^T} & -2\,\overline{p^T} & -\overline{p^T} & 0 & F_{13} \\ -2\,\overline{P} & -\overline{P} & 0 & 0 & F_{11} & F_{12} \\ F_1 & -A & 0 & 0 & F_2 & F_3 \\ -F_4 & (F_4 - A) & 0 & 0 & F_5 & F_6 \\ 0 & (B - F_7) & F_7 & -B & 0 & F_8 \\ 0 & B & -F_9 & (F_9 - B) & 0 & F_{10} \end{bmatrix}$$

and

$$v = \begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ p^* \\ p^T \end{bmatrix}$$

The symbols used in D are defined by

$$F_1 = 2\,c^2\,\overline{p}_1 \qquad F_7 = 2\,c^2\,\overline{p}_3 \qquad A = -\,R\Gamma^T\,\overline{P}/16$$

$$F_4 = 2\,c^2\,\overline{p}_2 \qquad F_9 = 2\,c^2\,\overline{p}_4 \qquad B = -\,R\Gamma^S\,\overline{p^T}/16$$

$$\overline{P} = (\overline{p^*} - \overline{p^T}) \qquad \kappa = R/c_p$$

$$F_2 = [-\,\overline{p}_1\,\overline{\alpha}_1\,c^2 + .75\,c^2\,(1-\kappa)\,\overline{\alpha}_1\,\overline{P} + .5\,A\,(3\,\overline{\alpha}_1 - \overline{\alpha}_2)]$$

$$F_3 = [\,\overline{p}_1\,\overline{\alpha}_1\,c^2 + .25\,c^2\,(1-\kappa)\,\overline{\alpha}_2\,\overline{P} + .5\,A\,(\overline{\alpha}_1 - 3\,\overline{\alpha}_2)]$$

$$F_5 = [-\,\overline{p}_2\,\overline{\alpha}_2\,c^2 + .25\,c^2\,(1-\kappa)\,\overline{\alpha}_2\,\overline{P} + .5\,A\,(3\,\overline{\alpha}_1 - \overline{\alpha}_2)]$$

$$F_6 = [\,\overline{p}_2\,\overline{\alpha}_2\,c^2 + .75\,c^2\,(1-\kappa)\,\overline{\alpha}_2\,\overline{P} + .5\,A\,(\overline{\alpha}_1 - 3\,\overline{\alpha}_2)]$$

$$F_8 = [-\,\overline{p}_3\,\overline{\alpha}_3\,c^2 + .75\,c^2\,(1-\kappa)\,\overline{\alpha}_3\,\overline{p^T} + .5\,B\,(3\,\overline{\alpha}_3 - \overline{\alpha}_4)]$$

$$F_{10} = [-\,\overline{p}_4\,\overline{\alpha}_4\,c^2 + .25\,c^2\,(1-\kappa)\,\overline{\alpha}_4\,\overline{p^T} + .5\,B\,(3\,\overline{\alpha}_3 - \overline{\alpha}_4)]$$

$$F_{11} = [4\,c^2 - .5\,\overline{P}(3\,\overline{\alpha}_1 + \overline{\alpha}_2)]$$

$$F_{12} = [-\,4\,c^2 - .5\,\overline{P}\,(\overline{\alpha}_1 + 3\,\overline{\alpha}_2)]$$

$$F_{13} = [4\,c^2 - .5\,\overline{p^T}\,(3\,\overline{\alpha}_3 + \overline{\alpha}_4)]$$

The value of the determinant was calculated for two, isothermal basic states ($\overline{T}$ = 250°K); in one case we used $\overline{p^T}$ = 200 mb and in the other $\overline{p^T}$ = 500 mb; $\overline{p^*}$ was 1000 mb in both cases. By interpolation, the roots of the determinant were estimated with the results:

	Isothermal $\overline{p^T}$ = 200 mb	Isothermal $\overline{p^T}$ = 500 mb
c_1	± 313 m sec^{-1}	± 311 m sec^{-1}
c_2	± 191 m sec^{-1}	± 134 m sec^{-1}
c_3	± 66 m sec^{-1}	± 42 m sec^{-1}
c_4	± 53 m sec^{-1}	± 21 m sec^{-1}

The isothermal, $\overline{p^T}$ = 500 mb, case corresponds most closely to the Phillips system and to the previously evaluated Shuman system two-layer model. Both c_1 and c_2 are quite close to the values calculated for the two-layer model. The values of c_1, c_3 and c_4 also agree closely with those obtained using Phillips four-layer model. Again there is a non-negligible difference between c_2 found in the Shuman and Phillips systems.

9. Summary and Conclusions

It has been determined that the fundamental (or fastest) free mode admitted in one, two and four layer models based upon Phillips or Shuman's σ systems are sensibly identical ($\approx$ 310 m sec^{-1}). The internal modes in the two layer model based on Phillips system was found to be considerably smaller than that found in the Shuman system (85 vs 130 m sec^{-1}). This distinction carried over to the four layer models but was modified somewhat because the Phillips system value of c_2 was increased to 110 m sec^{-1} apparently in reaction to the change in resolution of the basic state static stability. The slowest modes c_3 and c_4 were found to be essentially the same in the Phillips and Shuman system four layer models ($\simeq$ 43 and 24 m sec^{-1}).

The calculations reported herein were based upon isothermal basic states and must be used with caution in estimating the free modes in the general application of the models to real data.

Our principal conclusion is that the Shuman σ system's utilization of a free surface approximation for the "tropopause" has a dynamical significance for the free gravity modes. The possibility of utilizing this distinction in the design of semi-implicit integration schemes for the non-linear equations is therefore retained as a working hypothesis for our subsequent work.

10. References

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