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THE RELATIONSHIP OF THE FROUDE NUMBER TO NUMERICAL STABILITY OF THE GRAVITY WAVE EQUATIONS

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R. D. McPherson

Development Division

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to Numerical Stability of the Gravity Wave Equations

By R. D. McPherson

1. Introduction

In the course of a continuing series of numerical experiments concerning the computational stability of the gravity wave equations, in one space dimension,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + \phi \frac{\partial u}{\partial x} = 0 \tag{2}$$

and an auxiliary equation in the v-component of motion

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = 0, \tag{3}$$

evidence has been obtained that the onset of non-linear instability in (3) is somehow related to the initial distribution of the u velocity component. In particular, using the semi-momentum formulation for the spatial derivatives and the 'leapfrog' method for time differences, and initial velocity distributions given by

$$u_j = A \sum_{m=0}^{L/2} \sin \frac{2\pi mj\Delta x}{L\Delta x} \tag{4}$$

$$v_j = A \sum_{m=0}^{L/2} \cos \frac{2\pi mj\Delta x}{L\Delta x}, \tag{5}$$

instability was encountered at 85 days with A = 8.5, while with A = 2.18 the computations were quite well behaved at the nominal cutoff of 347 days. Here, instability is defined as occurring when the kinetic energy associated with the rotational component of the wind (v-component) exceeds its initial value by one order of magnitude.

In order to examine this behavior more carefully, eqns. (1-3) were non-dimensionalized, and a series of numerical calculations carried out as a function of the pertinent non-dimensional number. In this case, that number is the Froude number, defined as $F_r \equiv U/\sqrt{gH_o}$, where U is a characteristic fluid velocity.

2. Non-Dimensionalization of the Governing Equations

We define the independent non-dimensional variables

$$t' = t \left({}^{C_g}/L\Delta x \right) \tag{6}$$

and

$$x' = {}^{x}/L\Delta x \tag{7}$$

where $C_g = \sqrt{gH_o}$, H_o is the mean depth of the fluid, and L is the interval of periodicity of the initial data. We also require the following non-dimensional dependent variables:

$$\begin{aligned} u' &= {}^{u}/U \\ v' &= {}^{v}/U \\ \phi' &= {}^{\phi}/gH_o \quad , \end{aligned} \tag{8}$$

where U is a constant. With these definitions, (1) – (3) become

$$\frac{\partial u'}{\partial t'} + \frac{U}{C_g} u' \frac{\partial u'}{\partial x'} + \frac{C_g}{U} \frac{\partial \phi'}{\partial x'} = 0 \tag{9}$$

$$\frac{\partial \phi'}{\partial t'} + \frac{U}{C_g} \left(u' \frac{\partial u'}{\partial x} \right) = 0 \tag{10}$$

$$\frac{\partial v'}{\partial t'} + \frac{U}{C_g} u' \frac{\partial v'}{\partial x'} = 0. \tag{11}$$

Introducing the definition of the Froude number, and suppressing the prime notation with the understanding that all parameters are dimensionless, (9)-(11) become

$$\frac{\partial u}{\partial t} + F_r \; u \frac{\partial u}{\partial x} + F_r^{-1} \frac{\partial \phi}{\partial x} = 0 \tag{12}$$

$$\frac{\partial \phi}{\partial t} + F_r \left(u \frac{\partial u}{\partial x} + \phi \frac{\partial u}{\partial x} \right) = 0 \tag{13}$$

$$\frac{\partial v}{\partial t} + F_r \; u \frac{\partial v}{\partial x} = 0. \tag{14}$$

This system is transferred to a system of difference equations by use of the semi-momentum formulation for spatial derivatives and the leapfrog scheme for temporal differences:

$$\overline{u}^t_t + F_r \; \overline{\overline{u}^x \, u_x}^x + F_r^{-1} \; \overline{\phi}^x_x = 0 \tag{15}$$

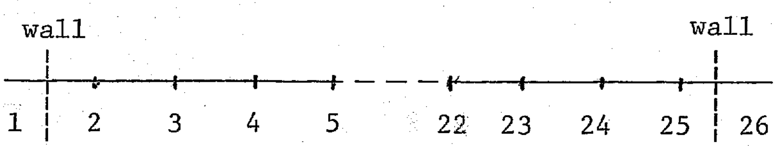
$$\overline{\phi}_t^{\,t} + F_r \left(\overline{\overline{u}^x \phi_x}^{\,x} + \overline{\overline{\phi}^x u_x}^{\,x} \right) = 0 \qquad (16)$$

$$\overline{v}_t^{\,t} + F_r \, \overline{\overline{u}^x v_x}^{\,x} = 0 \qquad (17)$$

This system of equations was integrated over a domain in which $L = 24$, Δx (dimensional) = 381 km, Δt (dimensional) = 600 sec., and $H_o = 7620$ m. These values were considered invariant in all experiments, and only F_r was varied; this means that only the characteristic wind speed was changed in each experiment. The initial velocity distribution was therefore computed from (4) and (5) where $A = (F_r)(C_g)$: as the Froude number increases, the magnitude of the winds increases, and the non-linear terms in the momentum equations (15) and (17) become relatively more important.

3. Boundary Conditions

It became apparent during the course of these and other one-dimensional numerical experiments that the specification of boundary conditions is very important to the success of the experiment. In the experiments reported here, it is assumed that there exist walls between the ultimate and penultimate grid points at either extremity of the lattice, as in the schematic:



grid point $i \rightarrow$

The appropriate boundary conditions are antisymmetry with respect to the u-component of velocity, and symmetry with respect to the v-component and the height field:

$$\overline{u}^x = v_x = h_x = 0 \left\{ \; i = 1\tfrac{1}{2},\ 25\tfrac{1}{2}. \right. \qquad (18)$$

In particular:

$$\begin{aligned} u_1 &= -u_2\,, & u_{26} &= -u_{25} \\ v_1 &= v_2\,, & v_{26} &= v_{25} \\ h_1 &= h_2\,, & h_{26} &= h_{25} \end{aligned} \qquad (19)$$

4. Results

Integrations were carried out initially for F_r = .05, .04, .03, .025, .02, .015, and .01. These correspond to an amplitude of the initial wind field of approximately 14, 11, 8, 7, 5.5, 4, and 3 m sec^{-1} respectively.

The results are displayed in Figure 1, which is a plot of the natural logarithm of RMS vorticity in ordinate against time in days in abscissa. The solid lines are for the above values of the Froude number.

It will be observed that the smallest value of F_r used yielded no growth in root-mean-square vorticity over 347 days, the terminus of the integration. On the other hand, with $F_r = .05$, rapid growth began at about 30 days, indicating the onset of instability. The intermediate values of F_r that were used showed an earlier onset of instability with increasing Froude number. This appears to indicate strongly that the onset of non-linear instability is closely related to the initial conditions.

However, this is not an unambiguous result. When other values of F_r were used (F_r = .027, .032, .037, 042), the results indicated that there is not a one-to-one relationship between the Froude number and the onset of instability. The dashed line gives the RMS vorticity for $F_r = .027$; the solution is apparently less badly behaved than that for $F_r = .025$.

Finally, an attempt was made to stabilize the calculations, in line with the Robert, Shuman, Gerrity theory. A simple filter was placed on the advecting coefficient in the v-equation, so that it becomes

$$\overline{v_t}^{t} + F_r \; \overline{\overline{u^*}^{x} \; v_x}^{x} = 0, \tag{20}$$

where $u^* = \frac{1}{2}\left(u^n + u^{n-1} \right)$. The result, for $F_r = .04$, is shown as the dashed line in Figure 2, whereas the solid line represents the unstabilized result extracted from Figure 1. The simple filter clearly has stabilized the solution.

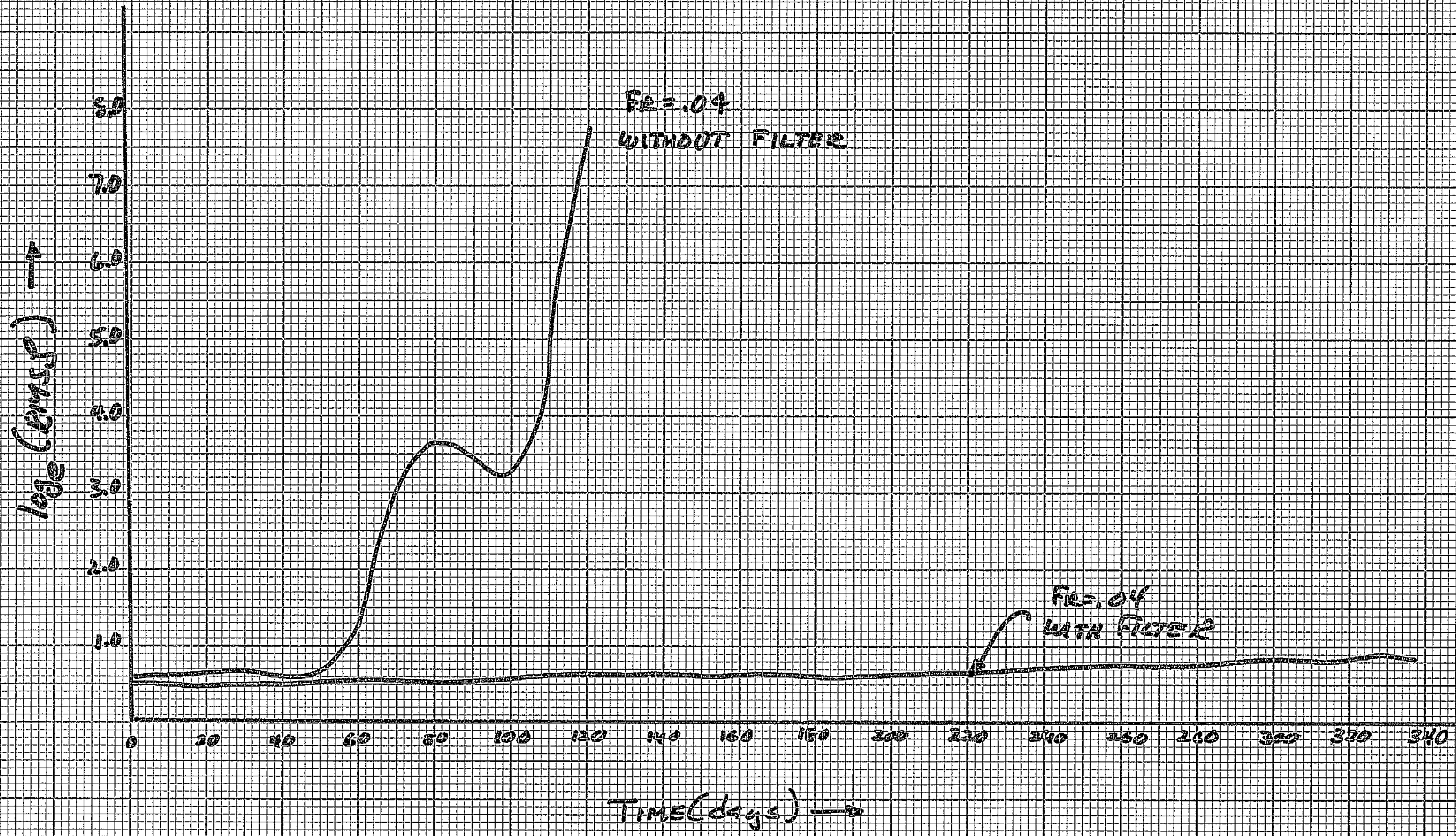

Fe=.04
WITHOUT FILTER
Fe=.04
WITH FILTER
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
TIME(days)

Figure 2.

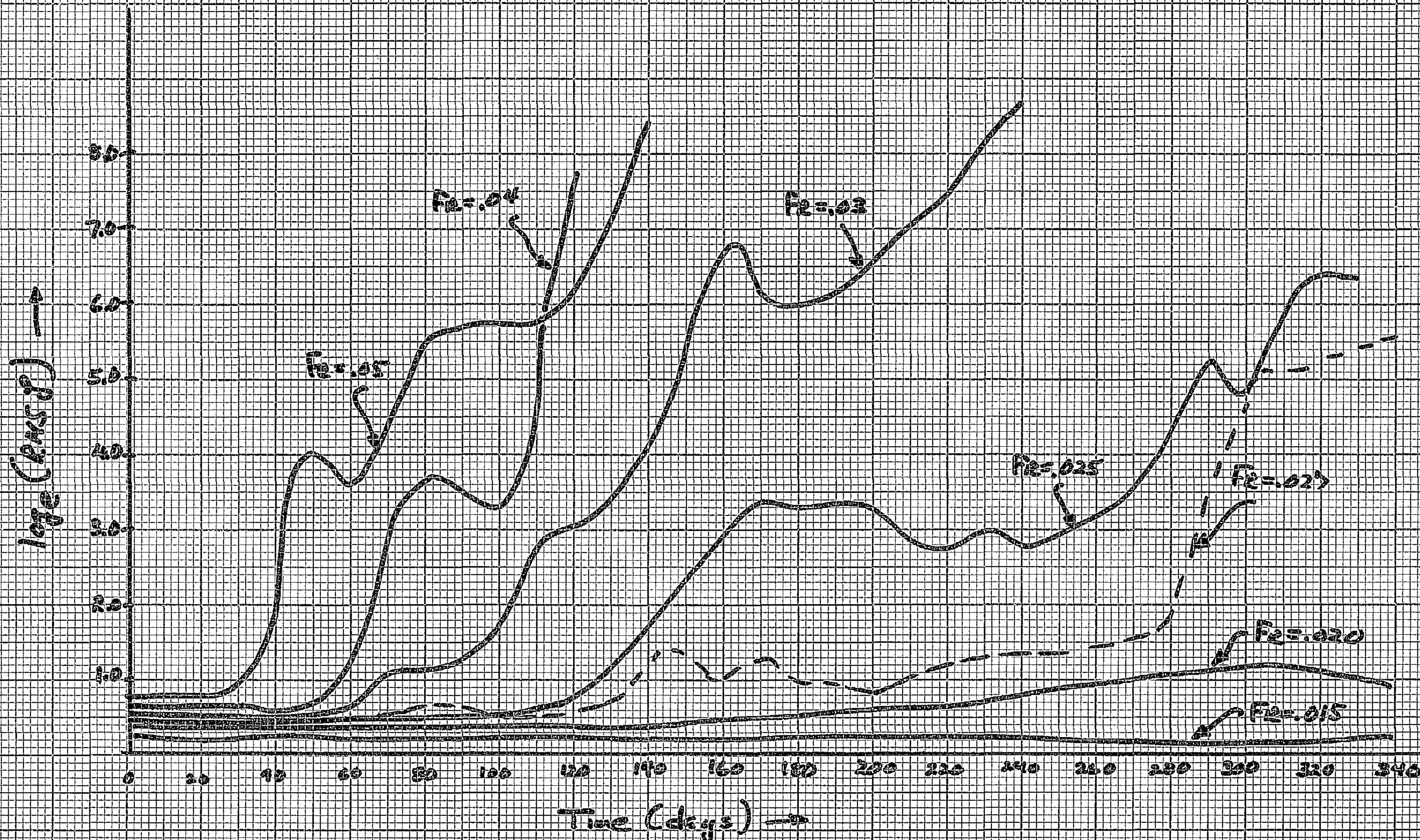

Fe=.04
Fe=.03
Fe=.045
Fe=.025
Fe=.027
Fe=.020
Fe=.015
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
Time (days)

Figure 1.