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OFFICE NOTE 15

ON THE DERIVATION OF THE OPERATIONAL BAROTROPIC SCALED FINITE-DIFFERENCE EQUATIONS

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JNWP OPERATIONAL MODELS
August 1958 to advent of IBM 7090 (1960)

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I. EVOLUTION OF JNWP MODELS

Routine forecasts were initiated at JNWP on 6 May 1955 using a 3-level model (see Charney (1.1)) covering an area somewhat larger than the United States. Due to the importance of boundary effects, efforts were made to extend the forecast area as much as possible within the limitations of the IBM 701 computer. The first step in this direction was the introduction of an experimental once daily barotropic forecast (see Charney and Phillips (1.2)) on September 29, 1955, covering approximately the area north of latitude 20 in the western hemisphere and north of latitude 50 in the eastern hemisphere. This was followed by introduction of the 2-level thermotropic model (see Thompson and Gates (1.3)) on 3 April 1956 covering an area intermediate between those of the 3-level and barotropic models. The results were so encouraging that on 2 July 1956 barotropic forecasts were increased to twice daily and the 3-level model was dropped completely in favor of the 2-level thermotropic model.

When the IBM 701 computer was deactivated on 3 June 1957, to make room for the IBM 704, only the barotropic forecast was continued during the transition period. The thermotropic model was never reprogrammed for the 704; instead a non-integrated 2-level model was developed. This was put into operation on 4 August 1958 and after 3 weeks was replaced by the current "mesh model"; so-called because it meshed the 500 mb barotropic forecast with the thickness forecast equation of the non-integrated 2-level model.

This model has the following characteristics:

a. The 500-mb barotropic forecast can be computed as a simple marching problem from one field of initial data and one tendency equation; thus there is no feed back from data at other levels at any stage. The dependent variable is the stream function on the 500 mb surface obtained initially by solving the balance equation by the method described by Shuman (1.4). The forecast equation is the vorticity equation modified as follows:

(1) The vertical advection of vorticity and the tilting terms are omitted.

(2) A stratospheric effect (i.e, Rossby free surface divergence and reduced gravity) is included to stabilize (i.e., prevent rapid retrogression) of the ultra-long waves.

(3) A term is included to approximate the generation of vorticity by terrain induced vertical motion.

b. The thickness tendency (or geostrophic thermal vorticity) equation contains the stream function of the mean motion as an independent variable and thus cannot be integrated as a simple marching problem. Mean level data must be supplied at each time step. As first ran, the 2-level

model contained tendency equations for both the thermal vorticity and the vorticity of the mean motion and had to be integrated as a marching jury problem - each equation requiring the output of the other at each time step. Since the mean level forecast obtained by this procedure was neither better than the barotropic 500 mb forecast in overall accuracy nor significantly different in the prediction of baroclinic effects, it was deactivated and the mean motion data required for the thickness forecast supplied from the barotropic output. The thickness forecast equation is the geostrophic thermal vorticity equation modified as follows:

(1) The vertical advection of vorticity and the tilting terms are omitted.

(2) The divergence of the thermal wind is approximated by $\frac{\omega_{max}}{\Delta p}$

(3) ω_m is in turn approximated from the isentropic thermodynamic equation.

II. THE BAROTROPIC MODEL

A. Derivation

Case studies by Vanderman and Drewes (2.1) and Arnason and Crestensen (2.2) indicate that the vertical advection of vorticity and tilting terms tend to be systematic and of opposite sign and to very nearly cancel each other over the whole field. Accordingly, unless both terms are retained neither should be retained. These findings were confirmed by a theoretical investigation by Wiin-Nielsen (2.3). Moreover, in finite difference form the vertical advection term is normally computed over a single grid distance and the tilting term over a double grid distance. This leads to inconsistent truncation between the two terms which introduces a systematic error into the vorticity of the mean motion when both terms are retained. Accordingly both terms have been dropped and the vorticity equation in the current operational model appears as

$$\frac{\partial \eta}{\partial t} + \mathbb{V} \cdot \nabla \eta - \eta \frac{\partial \omega}{\partial p} = 0 \tag{2.1}$$

The vertical velocity is considered to have both synoptic and terrain induced components. If we assume that the vertical velocity at the ground is due entirely to terrain we have

$$\omega_0 = \mathbb{V}_0 \cdot \nabla p_g \qquad (p_g = \text{ground pressure}) \tag{2.2}$$

producing divergence at the equivalent barotropic level of

$$-\frac{\partial \omega_0}{\partial p} \approx -\frac{\mathbb{V}_0 \cdot \nabla p_g}{p_0} \qquad (p_0 = 1000 \text{ mb}) \tag{2.3}$$

We now derive an expression for the synoptic component of the divergence following Wiin-Nielsen (2.4). We use the thermodynamic equation and equivalent barotropic assumptions.

$$(2.4) \qquad \mathbb{V}(p) = A(p)\,\mathbb{V}^*, \qquad \frac{dA}{dp} = \text{constant}$$

where the star denotes values at the equivalent barotropic level, assumed to be 500 mb.

This assumption implies that the thermal wind is parallel to the wind itself and thus there can be no temperature advection. Accordingly the thermodynamic equation reduces to

$$(2.5) \qquad \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + s\,\omega = 0, \qquad s \equiv -\alpha \frac{\partial \ln \theta}{\partial p}$$

Using for the moment the quasi-geostrophic approximation

$$(2.6) \qquad \mathbb{V} = \frac{1}{f}\,\mathbb{K} \times \nabla\phi = \frac{A}{f}\,\mathbb{K} \times \nabla\phi^*$$

$$(2.7) \qquad \frac{\partial \phi}{\partial p} = \frac{dA}{dp}\,\phi^*$$

and consequently

$$(2.8) \qquad \omega = -\frac{1}{s}\frac{dA}{dp}\frac{\partial \phi^*}{\partial t} = -\frac{f_0}{s}\frac{dA}{dp}\frac{\partial \psi^*}{\partial t}$$

Here the stream function, ψ^*, is defined in terms of the geopotential by

$$(2.9) \qquad \psi^* = \frac{\phi^*}{f_0}$$

(See Phillips (2.5))

To find $\frac{\partial \omega}{\partial p}$ from (8) only the variation of S with pressure is needed.

$$(2.10) \qquad S = -\alpha \frac{\partial \ln \theta}{\partial p} = \frac{RT}{p^2}\left(\frac{\gamma_d - \gamma}{\gamma_a}\right), \qquad \gamma_d \equiv \frac{g}{c_p}, \quad \gamma_a \equiv \frac{g}{R}$$

$$(2.11) \qquad \frac{\partial s}{\partial p} = -\frac{2RT}{p^3}\left(\frac{\gamma_d - \gamma}{\gamma_a}\right) - \frac{RT}{p^3}\left(\frac{\gamma_d - \gamma}{\gamma_a}\right)\frac{\gamma}{\gamma_a}$$

Since $\frac{\gamma}{\gamma_a} \simeq .188$ the second term is an order of magnitude smaller than the first. Thus to a good order of approximation, at least in the neighborhood of p*:

$$(2.12) \qquad S = s^*\left(\frac{p^*}{p}\right)^2, \qquad \frac{\partial s}{\partial p} = -\frac{2 s^* (p^*)^2}{p^3}$$

Accordinly

$$(2.13) \qquad \left(\frac{\partial \omega}{\partial p}\right)_{p^*} = \frac{f_0}{s^2}\frac{dA}{dp}\frac{\partial \psi^*}{\partial t}\frac{\partial s}{\partial p} = \frac{2 f_0}{s^* p^*}\left(-\frac{dA}{dp}\right)\frac{\partial \psi^*}{\partial t}$$

If we now assume that the divergent part of the horizontal wind is negligibly small so that the latter may be approximated in terms of a stream function and at the same time substitute (3) and (13) into (1) we obtain the prognostic equation

$$(2.14) \qquad (\nabla^2 - q\eta^*)\frac{\partial \psi^*}{\partial t} = -J(\psi^*, \eta^*) + a\eta^* J\left(\psi^*, \frac{p_s}{p_0}\right)$$

Here $q \equiv \frac{2 f_0}{s^* p^*}\left(-\frac{dA}{dp}\right)$ and $a = \left|\frac{V_0}{V^*}\right|$

Setting a = .2 in accordance with past practice

$$(2.15) \qquad -\frac{dA}{dp} = \frac{0.8}{500\ \text{mb}} = 1.6 \times 10^{-6} \text{ c.g.s.}$$

et $\gamma =$ 75% of the lapse rate of the standard atmosphere, then

$$(2.16) \quad s^* = \frac{R^2 T}{p^2 g}(\gamma_d - \gamma) = \frac{(\frac{2}{7}\times 10^7)^2 252}{25\times 10^{10}(980)}\left(\frac{9.96-5}{10^5}\right) = 4.13\times 10^{-4} \text{ c.g.s.}$$

$$(2.17) \quad q = \frac{2 f_0}{s^* p^*}\left(-\frac{dA}{dp}\right) = \frac{2(10^{-4})(1.6\times 10^{-6})}{(4.13\times 10^{-4})\,5\times 10^5} = 1.55\times 10^{-12} \text{ c.g.s.}$$

The so-called stratospheric term (the one containing q) of equation (14) was derived by Cressman (2.6) using Phillips' (2.7) tank model. Since the physics is more easily visualized in this derivation it is repeated below.

Consider two homogeneous and incompressible fluid layers of a densities ρ and $\rho' < \rho$, bounded by rigid horizontal plates at z = 0 and = H. For each fluid layer we may write the primitive equation of horizontal motion and the continuity equation as follows:

$$(2.18) \quad \mathbb{V}_t + \mathbb{V}\cdot\nabla\mathbb{V} + w\frac{\partial \mathbb{V}}{\partial z} + f\,\mathbb{K}\times\mathbb{V} = -\alpha\nabla p$$

$$(2.19) \quad \nabla\cdot\mathbb{V} + \frac{\partial w}{\partial z} = 0$$

We now adopt the quasi-static assumption, viz; the vertical acceleration is small and the balance of forces in the vertical may be expressed by the hydrostatic equation. Let $P(x,y,t) = \rho' g Z'$ be the pressure on the top of the upper fluid created by interaction between the fluid and the rigid plate at z = H and let $h(x,y,t)$ be the height of the interface between the two fluid layers. We then have in the upper and lower fluids respectively

$$(2.20) \quad \begin{cases} p'(x,y,z,t) = \rho' g Z' + \rho' g (H-z) \\ p(x,y,z,t) = \rho' g Z' + \rho' g (H-h) + \rho g (h-z) \end{cases}$$

$$(2.21)\quad \begin{cases} \alpha' \nabla p' = -g \nabla Z' \\ \alpha \nabla p = -\left[\frac{\rho'}{\rho} g \nabla Z' + g \frac{(\rho-\rho')}{\rho} \nabla h \right] = -g \nabla Z \end{cases}$$

where

$$(2.22)\quad Z \equiv \epsilon Z' + (1-\epsilon) h + (H-\delta), \qquad \epsilon \equiv \frac{\rho'}{\rho}$$

Z' and Z give the topography of pressure surfaces in the upper and lower fluids respectively. The topography of all surfaces in each layer are the same due to the assumptions of homogeneity and hydrostatic balance. Thus the horizontal velocity of each layer is independent of z and the continuity equation may be integrated directly

$$(2.23)\quad \begin{aligned} &\int_h^H \delta w' = -\int_h^H \nabla \cdot \mathbb{V}' \delta z \qquad && w'(h) = \frac{\partial h}{\partial t} + \mathbb{V}' \cdot \nabla h = (H-h) \nabla \cdot \mathbb{V} \\ &\int_0^h \delta w = -\int_0^H \nabla \cdot \mathbb{V} \delta z \qquad && w(h) = \frac{\partial h}{\partial t} + \mathbb{V} \cdot \nabla h = -h \nabla \cdot \mathbb{V} \end{aligned}$$

($\mathbb{K} \cdot \nabla \times$) operating on (18) yields the vorticity equation

$$(2.24)\quad \frac{\partial \zeta}{\partial t} + \mathbb{V} \cdot \nabla \eta + \eta \nabla \cdot \mathbb{V} = 0$$

Expressing the vorticity geostrophically:

$$(2.25)\quad \zeta = \frac{g}{f} \nabla^2 Z$$

(23), (24) and (25) give for the lower layer

$$(2.26)\quad \frac{g}{f} \nabla^2 \frac{\partial Z}{\partial t} + \mathbb{V} \cdot \nabla \eta - \frac{\eta}{h} \frac{dh}{dt} = 0$$

If the upper layer is considered to be inert, (22) yields

$$(2.27)\quad \frac{\partial h}{\partial t} = \frac{1}{(1-\epsilon)} \frac{\partial Z}{\partial t}$$

Adding to this the assumption that the motion of the lower fluid is parallel to the contours of the interface (i.e., $\mathbf{V}\cdot\nabla h = 0$) (26) becomes

$$(2.28) \qquad \frac{g}{f}\nabla^2\frac{\partial \zeta}{\partial t} + \mathbf{v}\cdot\nabla\eta - \frac{\eta}{(1-\epsilon)}\frac{1}{h}\frac{\partial \zeta}{\partial t} = 0$$

Comparing this with the non-divergent barotropic vorticity equation of a rigid top single fluid tank model, we see that two additional effects manifest themselves in (28). First, there is a term containing the divergence $\frac{1}{h}\frac{\partial Z}{\partial t}$, which arises because the upper surface of the active fluid (i.e., the tropopause in the atmospheric analogue) is allowed to vary. Secondly the magnitude of the divergence is amplified by the factor $\frac{1}{1-\epsilon} = \frac{\rho}{\rho-\rho'}$. This is due to the reduction in the gravitational restoring force on the perturbed interface compared to that which would be exerted if the upper fluid were not present, i. e., if $\rho' = 0$.

To incorporate this term into the non-geostrophic barotropic model, a stream function must again be introduced. An approximation similar to (9) is used.

$$(2.29) \qquad \psi = \frac{gZ}{f}$$

Substitution into (28) and adding the terrain effect from (14) yields

$$(2.30) \qquad \left(\nabla^2 - \frac{\mu}{\Psi}\eta\right)\frac{\partial\psi}{\partial t} = -J(\psi,\eta) + a\eta\, J\left(\psi, \frac{P_s}{P_0}\right)$$

in which the stratospheric term has been linearized by using a mean value of the stream function where it is undifferentiated and μ is a representative ratio defined by

$$(2.31) \qquad \mu \equiv \frac{Z}{(1-\epsilon)h}$$

Evaluationoof ϵ and consequently of μ from atmospheric sounding data appears doubtful. However, noting the correspondence

$$(2.32) \qquad \frac{\mu}{Z}\frac{\partial Z}{\partial t} = \frac{1}{h}\frac{\partial h}{\partial t}$$

where Z is the height of the 500 mb surface and h is the height of the tropopause, Cressman found a value of $\mu \approx 4$. He also made a series of forecasts from the same initial data with values of μ ranging from 0 to 8. These indicated an optimum value $\mu = 4$ with little change between $\mu = 4$ and $\mu = 8$.

Comparison of (30) and (14) reveals the correspondence

$$(2.33) \qquad \mu = q\bar{\Psi} = \frac{(1.55 \times 10^{-12})(980)(5.57 \times 10^{5})}{10^{-4}} \approx 8.5$$

Due to the numerous assumptions involved in this determination of μ, the empirically determined optimum value of $\mu = 4$ is still used.

B. Engineering of the Barotropic Equation

In binary computers multiplications and divisions by numbers other than integer powers of 2 consume approximately 10 times as much computational time as other operations. Further, keeping track of decimal points is greatly facilitated if all numbers used are expressed as a number less than unity times an integer power of 2. Accordingly in iterative

computations economy dictates that constant coefficients be reduced to a minimum by combination and by scaling of independent variables in units which absorb recurring coefficients other than integer powers of 2.

Constants and scaling used in the barotropic program (WP523) are as follows:

UNITS

$u = 4$

d = grid distance = 381 Km at 60°N

$d^2 = 14.5161 \times 10^{14}$ cm

$f = 1.45842 \times 10^{-4} \sin\phi \quad sec^{-1}$

$\bar{f} = 1.03125 \times 10^{-4} \quad sec^{-1}$

$g = 980$ cm sec^{-2}

$g/\bar{f} = 9.5114 \times 10^{6}$

$a = \frac{V_g}{V_{500}} = 0.2$

m = map scale $= \frac{1 + \sin 60°}{1 + \sin\phi}$

$\bar{\psi} = \frac{g}{\bar{f}}(18,280)(30.48) = 5.294543 \times 10^{12}$

SCALING

$$f = \hat{f}/4.5025 \times 10^2 \quad \text{or} \quad \hat{f} = .065665 \sin\phi$$

$$m^2 = 2.5898\ \hat{m}^2$$

$$m^2/d^2 = .17841 \times 10^{-14}\ \hat{m}^2$$

$$\psi = 2^{17} \frac{g}{\bar{f}} \hat{\psi} = 1.2467 \times 10^{12}\ \hat{\psi}$$

$$P_g/P_0 = 2.5\ \hat{p}$$

P_g = standard atmospheric pressure at smoothed topographic level

P_0 = 1000 mb

$$n = 22.24 \times 10^{-4}\ \hat{n}$$

FINITE DIFFERENCES

$$\frac{\partial}{\partial t} \equiv \frac{\Delta\tau}{7200} \qquad \Delta_\tau \psi \equiv \psi_{\tau+1} - \psi_{\tau-1}$$

$$\nabla^2 \equiv \frac{m^2}{d^2} \mathbb{V}^2$$

$$\mathbb{V}^2 \psi \equiv \psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{ij}$$

$$J \equiv \frac{m^2}{4d^2} \mathbb{J}$$

$$\mathbb{J}_{ij}(\psi, \eta) \equiv (\psi_{i+1,j} - \psi_{i-1,j})(\eta_{i,j+1} - \eta_{i,j-1}) - \\ -(\psi_{i,j+1} - \psi_{i,j-1})(\eta_{i+1,j} - \eta_{i-1,j})$$

$$\left(\frac{m^2}{d^2}\mathbb{V}^2 - \frac{4}{\bar{\psi}}\eta\right)\frac{\Delta_\tau \hat{\psi}}{7200} = \frac{m^2}{4d^2}\mathbb{J}(\eta, \hat{\psi}) + \frac{9m^2}{4d^2}\eta\,\mathbb{J}\left(\hat{\psi}, \frac{p_2}{p_0}\right)$$

$$\left[\mathbb{V}^2 - \frac{4(22.24\times10^{-2})}{(5.2945)(.17841)}\frac{\hat{\eta}}{\hat{M}^2}\right]\Delta_\tau\hat{\psi} = \frac{(7200)(22.24)10^{-4}}{4}\Big[\mathbb{J}(\hat{\eta}, \hat{\psi}) + \\ + .2\hat{\eta}\,\mathbb{J}(\hat{\psi}, 2.5\hat{p})\Big]$$

$$(2.34) \quad \left[\mathbb{V}^2 - .9421\frac{\hat{\eta}}{\hat{M}^2}\right]\Delta_\tau\hat{\psi} = 4.0032\,\mathbb{J}(\hat{\eta}, \hat{\psi}) + \\ + 2.0016\hat{\eta}\,\mathbb{J}(\hat{\psi}, \hat{p}).$$

The equation solved in the barotropic model at the present time is

$$(2.35)\quad \left[\nabla^2 - .944\,\frac{\hat{n}}{\hat{m}^2}\right]\Delta_\tau \hat{\psi} = 4J(\hat{n}, \hat{\psi}) + 2\hat{f}J(\hat{\psi}, \hat{p})$$

This is solved for successive 2-hour changes or tendencies of $\hat{\psi}$. For the initial iteration, $\Delta_\tau \hat{\psi}$ is divided by 2 and then added to the initial stream function, $\hat{\psi}_0$, to give a one hour forecast stream field. The forecast is then continued as long as desired by iteration of the formula

$$(2.36)\quad \hat{\psi}_{\tau+1} = \hat{\psi}_{\tau-1} + \Delta_\tau \hat{\psi}_\tau$$

The barotropic forecast is computed by the library program designated WP523. This program consists of a card deck and File #7 of the library tape, designated logical tape 1. The card deck feeds an initialization program, the initial 500 mb heights and stream function data, decision tables and code for printing the geostrophic isotachs. The library tape contains four records with contents as follows:

Record #1 contains 4040 words (7610_8)

0-1976 ($0-3670_8$) contains $\hat{f}$ in the LHW and $\hat{p}$ in the RHW.

1988-3960 ($3704_8 - 7574_8$) contains $\hat{m}^2$ in the LHW and zeros in the RHW.

Record #2 contains 1956 (3644_8) words constituting the utility programs and the main part of the barotropic program.

Record #3 contains (2162 (4162_8) words containing mainly the smoothing and print programs WP203, and 210.

Record #4 contains 2162 (4162_8) words containing mainly WP9.

In writing the barotropic program, the data decks used to supply $\hat{f}$ and $\hat{p}$ contained these fields in the units:

$$\hat{f}' = .524942 \sin\varphi, \qquad \hat{p}' = \frac{1}{2}\left[\frac{P_0(mb)}{2^{14}}\right]$$

These were then multiplied respectively by the constants .12509 and .8192 thus obtaining $\hat{f}$ and $\hat{p}$ as defined above.

(To counteract the non-conservation or loss of vorticity due to iterated truncation the stream function is multiplied by the factor 1.13 before the barotropic forecasts begins. This factor is removed before outputting stream function data for relaxation, history tapes and punching.)

The 500-mb stream function read onto tapes 2 and 4 initially and at the end of each time step contains the coefficient L = 0.741252952 which is not otherwise used by the program.

To avoid handling of large numbers all pressure height and stream function data are punched and handled in the computer as departures from normal. Appropriate constants are inserted in the print programs to obtain printouts of absolute values.

III. THE 2-LEVEL MODEL

A. Derivation

An atmospheric model is assumed in which:

(1) The wind varies linearly with pressure. (Note that this is the thermotropic rather than the equivalent barotropic assumption).

(2) The vertical velocity has a maximum at the level of non-divergence, assumed to be at 600 mb and vanishes at 200 and 1000 mb. The variation of ω with pressure between these levels is at most quadratic in p.

(3) The termal wind and thermal vorticity are quasi-geostrophic.

(4) The vorticity equation may be approximated by equation (2.1).

Equation (2.1) is now applied at 400 mb identified by subscript 1 and 800 mb identified by subscript 3 and the latter equation subtracted from the former yielding

$$(3.1) \qquad \frac{\partial}{\partial t}(\eta_1 - \eta_3) + \mathbb{V}_1 \cdot \nabla \eta_1 - \mathbb{V}_3 \cdot \nabla \eta_3 - \left(\eta_1 \frac{\partial \omega_1}{\partial p} - \eta_3 \frac{\partial \omega_3}{\partial p}\right) = 0$$

Using modeling assumption (1) we define bar and prime quantities as follows:

$$(3.2) \qquad \overline{(\ )} \equiv \frac{(\ )_1 + (\ )_3}{2} = (\ )_2, \quad \text{i.e. values}$$

at level 2 midway between levels 1 and 3.

(3.3) $$(\ )' \equiv \frac{(\ )_1 - (\ )_3}{2}$$

Equation (1) now takes the form

(3.4) $$\frac{\partial \eta'}{\partial t} + \bar{\mathbb{V}} \cdot \nabla \eta' + \mathbb{V}' \cdot \nabla \bar{\eta} - \frac{1}{2}\left(\eta_1 \frac{\partial \omega_1}{\partial p} - \eta_3 \frac{\partial \omega_3}{\partial p}\right) = 0$$

We can evaluate $\frac{\partial \omega}{\partial p}$ by the finite difference ratios.

(3.5) $$\frac{\partial \omega_1}{\partial p} = \frac{\omega_m - \omega_0}{p_2 - p_0} = \frac{\omega_m}{\Delta p} \qquad \Delta p \equiv 400 \text{ mb}$$
$$\frac{\partial \omega_3}{\partial p} = \frac{\omega_4 - \omega_m}{p_4 - p_2} = -\frac{\omega_m}{\Delta p}$$

Here ω is placed under the Δ to identify the origin of this term. (4) now simplifies to

(3.6) $$\frac{\partial \eta'}{\partial t} + \bar{\mathbb{V}} \cdot \nabla \eta' + \mathbb{V}' \cdot \nabla \bar{\eta} - \bar{\eta} \frac{\omega_m}{\Delta p} = 0$$

It is apparent from (5) that $\bar{\mathbb{V}}$ is non-divergent and may be expressed in terms of a stream function. Also, using modeling assumption (3), we can express the thermal wind and vorticity in terms of the thickness.

(3.7) $$\bar{\mathbb{V}} = \mathbb{K} \times \nabla \bar{\Psi}$$

(3.8) $$\mathbb{V}' = \frac{g}{2f} \mathbb{K} \times \nabla h \qquad \zeta' = \frac{g}{2f} \nabla^2 h$$

Introducing these in (6) we obtain

(3.9 $$\nabla^2 \frac{\partial h}{\partial t} + f J\left(\bar{\Psi}, \frac{1}{f} \nabla^2 h\right) + J(h, \bar{\eta}) - \frac{2f \omega_m}{g \Delta p} \bar{\eta} = 0$$

We now eliminate ω_m using the adiabatic thermodynamic equation

$$\frac{d\theta}{dt} = 0 = \frac{d\ln\theta}{dt} = \frac{\partial \ln\theta}{\partial t} + \mathbb{V}\cdot\nabla\ln\theta + \omega\frac{\partial\ln\theta}{\partial p} \tag{3.10}$$

$$\theta \equiv T\left(\frac{1000}{p}\right)^{\frac{R}{c_p}} = \frac{p\alpha}{R}\left(\frac{1000}{p}\right)^{\frac{R}{c_p}} \tag{3.11}$$

$$\ln\theta = \ln\alpha + \left(1-\frac{R}{c_p}\right)\ln p + \text{constant} \tag{3.12}$$

Since we are working in pressure corrdinates the first two terms on the right of (10) are computed from (12) holding the pressure constant. Multiplying through by the specific volume, α, (10) becomes

$$0 = \frac{\partial\alpha}{\partial t} + \mathbb{V}\cdot\nabla\alpha - s\omega, \qquad s \equiv -\alpha\frac{\partial\ln\theta}{\partial p} \tag{3.13}$$

Evaluating α at the mid-level using a finite difference ratio yields

$$\bar{\alpha} = -g\left(\frac{\partial\bar{z}}{\partial p}\right) \approx -g\,\frac{z_3 - z_1}{p_3 - p_1} = \frac{gh}{\Delta_h p} \tag{3.14}$$

$$\omega_m = \frac{g}{s\Delta_h p}\left(\frac{\partial h}{\partial t} + \bar{\mathbb{V}}\cdot\nabla h\right) \tag{3.15}$$

Substitution of this in (9) leads to the prognostic thickness equation giving the thickness tendency in terms of space derivatives only of the initial fields of 800 to 400 mb thickness and 600 mb stream functions.

$$\left(\nabla^2 - \frac{2f\bar{n}}{\sigma^2}\right)\frac{\partial h}{\partial t} = \frac{2f\bar{n}}{\sigma^2}J(\bar{\Psi}, h) - fJ\left(\bar{\Psi}, \frac{1}{f}\nabla^2 h\right) - J(h, \bar{n}) \tag{3.16}$$

here

(3.17) $$\sigma^{2\,\circ} \equiv s\,\hat{\omega}p\,\hat{h}p$$

To complete the system of predictive equations, an equation for $\frac{\partial \bar{\Psi}}{\partial t}$ is required. Such an equation can be obtained either by applying equation (2.1) at levels 1 and 3 and adding or by applying it directly to level 2. The latter procedure leads immediately to the non-divergent barotropic vorticity equation

(3.18) $$\nabla^2 \frac{\partial \bar{\Psi}}{\partial t} = - J(\bar{\Psi}, \nabla^2\bar{\Psi} + f)$$

The former procedure leads to

(3.19) $$\nabla^2 \frac{\partial \bar{\Psi}}{\partial t} = - J(\bar{\Psi}, \nabla^2\bar{\Psi} + f) - \frac{g^2}{4f} J(h, \tfrac{1}{f}\nabla^2 h) + \frac{g\,\omega_m}{2f\,\hat{\omega}p}\nabla^2 h$$

which contains two additional terms; the Sutcliffe development term and a divergence term. The Sutcliffe development term, the second on the right of equation (19) is highly correlated with the term preceding it. In fact, in the equivalent barotropic model it appears simply as an increase of approximately 25% in the coefficient of the vorticity advection term, $J(\bar{\Psi}, \nabla^2\bar{\Psi} + f)$ the divergence term, the third term on the right of equation (19), contains two quantities which are correlated, i.e. ω_m and $\nabla^2 h$. Thus, this term tends to have the same sign over the whole field so that repeated iterations lead to a net drop in heights over the whole field with a maximum in the center

Accordingly equation (19) was applied in the form

$$(3.20) \quad \nabla^2 \frac{\partial \bar{\Psi}}{\partial t} = - J(\bar{\Psi}, \nabla^2 \bar{\Psi} + f) - \frac{g^2}{4f} J(h, \tfrac{1}{f} \nabla^2 h)$$

For reasons mentioned above it was considered unlikely that the feed back from the thickness field expressed by the Sutcliffe term would produce a forecast significantly different from that already being produced by the operational barotropic model.

This belief was confirmed by a 2 1/2 week trial period in August 1958 during which forecasts were prepared both with the 2-level model using equations (16) and (20) and with the operational barotropic model using (2.30). While this result was disappointing from the development point of view, it was operationally advantageous in that the cheapest 500 mb forecast also proved to be at least as good as any available. Accordingly use of equation (20) was discontinued and the current "mesh model" initiated.

B. Modifications resulting from changes in levels of input data

Since 800 mb is not a standard data level, the 2-level model was originally planned to use 850 and 400 mb data. For reasons of economy 850 and 500 md data were actually used. With the thermotropic modeling assumption that the wind varies linearly with pressure or that the thermal gradient is independent of pressure, any level of input data can be used.

Equation (16) is linear in h and thus holds regardless of the thickness layer used so long as the value of $\frac{\Delta p}{h}$ in σ^2 is consistent with the model

Rewrite (14) using subscripts i for input and m for model levels

(3.21) $$\alpha_m = \frac{g h_m}{\Delta_h p_m} = \frac{R T_m}{p_m}, \quad \alpha_i = \frac{g h_i}{\Delta_h p_i} = \frac{R T_i}{p_i}$$

(3.22) $$T_i = T_m + \delta$$ (δ = constant - thermotropic model)

(3.23) $$\alpha_i = \frac{R(T_m + \delta)}{p_i}$$

(3.24) $$\alpha_m = \frac{T_m p_i}{(T_m + \delta) p_m} \alpha_i = \frac{T_m}{T_m + \delta} \frac{p_i}{p_m} \frac{g h_i}{\Delta_h p_i}$$

(3.25) $$h_m = \frac{T_m}{T_m + \delta} \frac{p_i}{p_m} \frac{\Delta_h p_m}{\Delta_h p_i} h_i \equiv K(i,m) h_i$$

Substitute in (15) and (16). In (16) the coefficient h_i appears in every term and thus leaves the equation unchanged. From (15)

(3.26) $$\omega_m = \frac{g}{s_m \Delta_h p_m} \left(\frac{\partial h_i}{\partial t} + \bar{V} \cdot \nabla h_i \right) K(i,m)$$

Thus regardless of the levels of input data, the only change in the equations will be the coefficient $K(i,m)$ in the ω -equation.

However, input data levels which most accurately represent the vertical wind shear and horizontal temperature gradient of the entire troposphere should give best results. Thus presumably the greater Δp the better so long as data levels do not lie about the level of maximum wind or tropopause or so close to the ground that the data are influenced by surface effects. From this it again appears that input data level near 800 and 400 mb would be best.

C. Engineering of the thickness forecast equation

Constants and scaling used in the thickness forecast program (WP566) are as follows:

UNITS

d = grid distance = 381 km at 60° N

$d^2 = 14.5161 \times 10^{14}$ cm^2

$f = 1.45842 \times 10^{-4} \sin\phi$ sec^{-1}

$\bar{f} = 1.03125 \times 10^{-4}$ sec^{-1}

$g/\bar{f} = 9.503 \times 10^{6}$

m = map scale factor = $\dfrac{1 + \sin 60^\circ}{1 + \sin\phi}$

$\bar{\psi} = .75\ \psi_{500}$

FIRST SCALING

$$h = 2^{17}\hat{h}$$

$$\psi = 2^{17}\frac{g}{f}\hat{\psi}$$

$$n = 2^{-9}\hat{n} \qquad (\text{insures } \hat{n} < 1 \text{ for } n \leq 8f)$$

$$m = 2\hat{m} \qquad (\text{insures } \hat{m} \leq 1)$$

$$m^2/d^2 = .275556 \times 10^{-14}\,\hat{m}^2$$

$$\sigma^2 = 2^{28}\hat{\sigma}^2$$

FINITE DIFFERENCES

$$\frac{\partial}{\partial t} = \frac{\Delta\tau}{3600} \qquad \Delta_\tau h = \tfrac{1}{2}\left(h_{\tau+1} - h_{\tau-1}\right)$$

$$\nabla^2 = m^2/d^2\ \mathbb{V}^2$$

$$\mathbb{V}^2 h = h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - 4h_{i,j}$$

$$J = \frac{m^2}{4d^2}\mathbb{J}$$

$$\mathbb{J}(h;n) = (h_{i+1,j} - h_{i-1,j})(n_{i,j+1} - n_{i,j-1}) -$$

$$- (h_{i,j+1} - h_{i,j-1})(n_{i+1,j} - n_{i-1,j})$$

$$\hat{n} = 2^{9}\left[2^{17}(.75)\frac{g}{f}\frac{m^2}{d^2}\mathbb{V}^2\hat{\psi} + 1.45842\times 10^{-4}\sin\phi\right]$$

$$= 2(.33554432)(.75)(9.503)(.275556)\,\hat{m}^2\mathbb{V}^2\hat{\psi} + .0746711\sin\phi$$

$$= 2(.658947)\,\hat{m}^2\mathbb{V}^2\hat{\psi} + .0746711\sin\phi$$

Substitution in (16) yiedls:

$$\left(\frac{4\hat{m}^2}{d^2}\nabla^2 - \frac{2f\hat{n}\,2^{-9}}{2^{28}\hat{\sigma}^2}\right)\frac{\Delta_t\hat{h}}{3600} = -\sin\phi\, 2^{17}\frac{g}{\bar{f}}\frac{4\hat{m}^2}{4d^2}(.75)\,J\left(\hat{\psi}, \frac{4\hat{m}^2}{d^2\sin\phi}\nabla^2\hat{h}\right)$$

$$-\frac{4\hat{m}^2}{4d^2}2^{-9}J(\hat{h},\hat{n}) + \frac{2f\hat{n}\,2^{-9}}{2^{28}\hat{\sigma}^2}2^{17}\frac{g}{\bar{f}}(.75)\frac{4\hat{m}^2}{4d^2}J(\hat{\psi},\hat{h})$$

$$\left(\nabla^2 - \frac{fd^2\hat{n}}{2^{38}\hat{m}^2\hat{\sigma}^2}\right)\Delta_t h = -3600\sin\phi(.75)(2^{17})\frac{g}{\bar{f}d^2}J\left(\hat{\psi}, \frac{\hat{m}^2}{\sin\phi}\nabla^2\hat{h}\right)$$

$$-2^{11}(3600)J(\hat{h},\hat{n}) + 3600(2^{-21})\frac{fg}{\bar{f}}(.75)\frac{\hat{n}}{\hat{\sigma}^2}J(\hat{\psi},\hat{h})$$

$$\frac{fd^2\hat{n}}{2^{38}\hat{m}^2\hat{\sigma}^2} = \frac{(1.45842)(14.5161)10^{10}\hat{n}\sin\phi}{27.48779\times10^{10}\hat{m}^2\hat{\sigma}^2} = .77018\,\frac{\hat{n}\sin\phi}{\hat{m}^2\hat{\sigma}^2}$$

$$2^{17}(.75)(3600)\frac{g}{\bar{f}d^2} = \frac{2^2(3.6)(9.503)(.75)(.32768)}{14.5161} = 2^2(.579195)$$

$$2^{-11}(3600) = \frac{2(900)}{1024} = 2(.87890625)$$

$$3600(.75)(2^{-21})\frac{fg}{\bar{f}} = \frac{2^2(9.503)(.75)(1.45842)(.9)\sin\phi}{20.97152} = 2^2(.446085)\sin\phi$$

$$\left(\nabla^2 - .7702\frac{\hat{n}\sin\phi}{\hat{m}^2\hat{\sigma}^2}\right)\Delta_t\hat{h} = -2^2(.5792)\sin\phi\, J\left(\hat{\psi}, \frac{\hat{m}^2}{\sin\phi}\nabla^2\hat{h}\right)$$

$$-2(.8789)J(\hat{h},\hat{n}) + 2^2(.4461)\sin\phi\frac{\hat{n}}{\hat{\sigma}^2}J(\hat{\psi},\hat{h})$$

The first scaling was established to prevent spilling. We now introduce a second scaling to reduce the number of multiplicative constants which are not integer powers of 2.

SECOND SCALING

(3.27) Let $\overset{*}{\hat{n}} = .878906\,\hat{n} = 2(.579152)\hat{m}^2\nabla^2\hat{\psi} + .065629\sin\phi$

This will eliminate the fractional coefficient of the second term on the right. Substitution yields:

$$\left(\nabla^2 - \frac{.7702\,\overset{*}{\hat{n}}\sin\phi}{.8789\,\hat{m}^2\hat{\sigma}^2}\right)\Delta_c\hat{h} = -4(.5782)\sin\phi\,J\left(\hat{\psi}, \frac{\hat{m}^2}{\sin\phi}\nabla^2\hat{h}\right)$$

$$-2J(\hat{h}, \overset{*}{\hat{n}}) + 4\left(\frac{.4461}{.8789}\right)\sin\phi\,\frac{\overset{*}{\hat{n}}}{\hat{\sigma}^2}J(\hat{\psi}, \hat{h})$$

We can now absorb the coefficient of the last term on the right by rescaling $\hat{\sigma}^2$ as

(3.28) $$\overset{*}{\hat{\sigma}}{}^2 = \frac{.8789}{.4461}\hat{\sigma}^2 = 1.9702\,\hat{\sigma}^2 = 2^{-28}(1.9702)\,\sigma^2$$

$$\left(\nabla^2 - \frac{(.7702)(1.9702)}{.8789}\,\frac{\overset{*}{\hat{n}}\sin\phi}{\hat{m}^2\overset{*}{\hat{\sigma}}{}^2}\right)\Delta_c\hat{h} = -4(.5792)\sin\phi \times$$

$$J\left(\hat{\psi}, \frac{\hat{m}^2}{\sin\phi}\nabla^2\hat{h}\right) - 2J(\hat{h}, \overset{*}{\hat{n}}) + 4\sin\phi\,\frac{\overset{*}{\hat{n}}}{\overset{*}{\hat{\sigma}}{}^2}J(\hat{\psi}, \hat{h})$$

We can now write equations (27) and (28) as

$$\hat{\overset{*}{n}} = 2A\hat{m}^2\nabla^2\hat{\psi} + B\sin\phi \tag{3.27}$$

$$\left(\nabla^2 - C\frac{\hat{\overset{*}{n}}\sin\phi}{\hat{m}^2\,\hat{\overset{*}{g}}^2}\right)\Delta_\tau\hat{h} = -4B\sin\phi\,J\left(\hat{\psi}, \frac{\hat{m}^2}{\sin\phi}\nabla^2\hat{h}\right) - 2J(\hat{h}, \hat{\overset{*}{n}}) + 4\sin\phi\,\frac{\hat{\overset{*}{n}}}{\hat{\overset{*}{g}}^2}J(\hat{\psi}, \hat{h}) \tag{3.28}$$

where

A = .5792

B = .065629

C = .8633

D. The Stability factor and vertical motion

The quantity s defined by equation (29) conveniently represents the convective stability of dry air. In the development of most NWP models constants or quasi-constants such as f and Δp are lumped with s and the combined term is christened the stability factor. Thus, the term stability factor has no commonly accepted definition. In this note the term

$$s \equiv -\alpha\frac{\partial \ln\theta}{\partial p} = \frac{RT}{p^2}\left(\frac{\gamma_d - \gamma}{\gamma_a}\right) = \frac{1}{\rho T}\left(\frac{1}{c_p\rho} - \frac{\partial T}{\partial p}\right) \tag{3.29}$$

will be called stability or stability coefficient term and the term

$$\sigma^2 \equiv s\,\Delta p\,\Delta p \tag{3.28}$$

will be called stability factor. The stability is a function of $x, y, p, \& t$ but it varies most strongly with pressure. In 2-level models S is usually treated as a constant but need not be. The variations of S may be approximated through the thermodynamic equation as Thompson (3.1) did or by relating S to $\bar{\psi}$ as Vanderman is now trying in applying equation (16) to 500 and 200 mb data.

In Table 1 are tabulated values of S for the standard atmosphere computed from the three formulas indicated. The finite difference ratios presumably give mean values of S for the layers concerned.

TABLE 1

Values of the stability coefficient $S = -\alpha \frac{\partial \ln \theta}{\partial p}$ computed from the standard atmosphere as point values and over the layer 800-400 mb using the formulas indicated.

Pressure level (mb)	S in 10^{-4} c.g.s. units		
	$S = \frac{RT}{p^2}\left(\frac{\gamma_d - \gamma}{\gamma_a}\right)$	$S = \frac{1}{p\theta}\frac{\Delta\theta}{\Delta p}$	$S = \frac{1}{pT}\left(\frac{1}{c_p p} - \frac{\Delta T}{\Delta p}\right)$
100	179		(800-400 mb level using 600 mb values of $p, \theta, \& T$)
200	44.8		
300	6.96		
350	5.26		
400	4.13		
500	2.76		
600	1.98	2.07	1.91
675	1.60		
700	1.50		
800	1.18		
850	1.06		
900	.954		
1000	.787		

It is apparent from (17) that σ^2 is not simply a function of

stability since it also contains $\Delta_s p$ and $\Delta_h p$. These are fixed by the modeling assumptions at 400 mb each. Accordingly we obtain the values in Table 2 for σ^2 from the values of s in Table 1.

TABLE 2

Values of σ^2 (in 10^8 c.g.s. units) using the values of from Table 1 and $\Delta_s p = \Delta_h p = 400$ mb

$S = \frac{RT}{p^2}\left(\frac{\gamma_d - \gamma}{\gamma_a}\right)$	$S = \frac{1}{p\theta}\frac{\Delta\theta}{\Delta p}$	$S = \frac{1}{pT}\left(\frac{1}{c_p p} - \frac{\Delta T}{\Delta p}\right)$
.317	.332	.306

In his original derivation of the 2-level model Thompson (3.1) obtained a σ^2-equation. Since he planned to use 850 and 400 mb input data he defined σ^2 in terms of these levels and let $\Delta_s p = \Delta_h p$ = 450 mb. Revised computations of the stability factor have continued this practice.

The most recent revision by Captain John A. Brown, USAF, computes summer and winter values from the equation

$$\sigma^2 = \frac{\Delta_s p \, \Delta_h p}{pT}\left(\frac{1}{c_p p} - \frac{\Delta T}{\Delta p}\right) = \frac{\Delta p}{pT}\left(\frac{\Delta p}{p c_p} - \Delta T\right) \tag{3.30}$$

where $\Delta_s p = \Delta_h p = \Delta p = 850 - 400 = 450$ mb.

$$\left.\begin{array}{l}\rho = .8 \times 10^{-3} \\ T = 261\end{array}\right\} \text{600 mb standard atmosphere values.}$$

$$\Delta T = T_{850} - T_{400} = 30^\circ C \text{ in winter}$$

$$= 36^\circ C \text{ in summer}$$

$$= 37^\circ C \text{ (standard atmosphere)}$$

$$\sigma^2 = \frac{4.5 \times 10^5}{(261)(.8 \times 10^{-3})}\left(\frac{4.5 \times 10^5}{.8 \times 10^4} - \Delta T\right) = 2.156 \times 10^6 (56.2 - \Delta T)$$

$$\sigma^2(\text{summer}) = .435 \times 10^8$$

$$\sigma^2(\text{winter}) = .564 \times 10^8$$

These values are considerably larger than those in Table 2. Part of the increase in the winter value is due to an actual increase in the stability of the atmosphere but both values are considerably larger than those of Table 2. due to the larger value of Δp. This can be considered in two ways - first as a change in the assumed ω-profile or secondly as an artificial increase in static stability. In either case the effect of the larger value is to reduce the effect of the divergence or vertical motion term in the thickness equation and thus generally to reduce the magnitude of the thickness tendencies.

The ω-equation is corrected for the effect of the larger $\Delta p_s'$ by virture of containing Δp explicitly and by a multiplicative constant 1.42

representing the ratio of 850-400 thickness to the 850-500 thickness. These corrections, however, cannot correct for the effect of σ^2 on the thickness tendencies which in turn influence ω.

We may write the corrected ω-equation as

$$\omega = 1.42 \frac{g \Delta p}{\sigma^2}\left(\frac{\partial h}{\partial t} + \bar{V}\cdot\nabla h\right)$$

$$= \frac{(1.42)(980)(4.5\times10^5)}{2^{28}\,\frac{\hat{\sigma}^{*2}}{1.9702}}\left[\frac{2^{17}\Delta_\tau \hat{h}}{3600} + \frac{m^2}{4d^2} J\left(.75\frac{g}{f}2^{17}\hat{\psi},\ 2^{17}\hat{h}\right)\right]$$

$$= \frac{2\times10^2(1.9702)(1.42)(.98)(4.5)}{(3.6)(4.096)\,\hat{\sigma}^{*2}}\left[\Delta_\tau \hat{h} + 2^{17}(.9)(.75)(9.503)(.275556)\,\hat{m}^2 J(\hat{\psi},\hat{h})\right]$$

or

$$(3.31)\quad \omega = \frac{2(.83671)10^2}{\hat{\sigma}^{*2}}\left[\Delta_\tau \hat{h} + 4(.5792)\,\hat{m}^2 J(\hat{\psi},\hat{h})\right]$$

Let $\omega = 2^7\hat{\omega}$

$$\hat{\omega} = \frac{2(.83671)}{1.28\,\hat{\sigma}^{*2}}\left[\Delta_\tau \hat{h} + 4(.5792)\,\hat{m}^2 J(\hat{\psi},\hat{h})\right]$$

or

$$(3.32) \qquad \hat{\omega} = \frac{2D}{\hat{\gamma}^{*2}} \left[\Delta_\tau \hat{h} + 4A\hat{m}^2 J(\hat{\psi}, \hat{h}) \right]$$

A = .5972

D = .6539

Equation (32) is the one from which vertical velocity is computed by the mesh model. The ω-maps printed by the program print tens, units and tenths of the negative of units of eq. (31), i.e., $-$ dynes cm^{-2} sec^{-1} at every other grid point with contours at intervals of 1.5 units.

Since the computed value of omega is for the 600 mb level, we can obtain the equivalence in cm sec^{-1} by taking the local derivative on the 600-mb surface of the hydrostatic equation

$$(3.33) \qquad w = -\frac{\omega}{\rho g} = -\frac{\omega}{.8 \times 10^{-3} \times 980} = -1.28\,\omega$$

Thus the printed ω-maps are in units of .78 cm sec^{-1} at 600 mb.

4. 7090 VERSION OF THE MESH MODEL

On 8 July 1960, the IBM 704 computer was deactivated to make room for an IBM7090. During the transition period the mesh model was run operationally on the National Bureau of Standards' 704 computer.

On 12 September 1960, routine operations were initiated on the 7090 with a new version of the mesh model. Other than programming changing this model differs from the 704 version in the following respects:

a. The mountain term has been improved and a surface friction term has been added. See Cressman (4.1).

b. The factor 1.13 described on page 15 for counteracting iterative truncation of the vorticity field now multiplies only the vorticity advection term.

d. Stream field data output to the history tape no longer contains the factor L = 0.741252952.

The 7090 mountain term treats the ground pressure and wind speed as variables throughout the term and assumes that the terrain induced vertical motion is completely compensated by horizontal divergence in the troposphere, i.e., below 200 mb. Equations (2.2) and (2.3) are now expressed as

$$\omega_0 = \omega_g = \mathbb{V}_g \cdot \nabla P_g \tag{4.1}$$

$$-\frac{\partial \omega_g}{\partial p} \approx -\mathbb{V}_g \cdot \nabla P_g / (P_g - 2 \times 10^5) \tag{4.2}$$

The constant $a = \frac{V_g}{V_{500}}$ is replaced by the variable defined as follows:

$$r = \frac{V_g}{V_{500}} = 1 - 0.8\left(\frac{P_g - 5\times10^5}{5\times10^5}\right) = 1.8 - 1.6\, P_g \times 10^{-6} \tag{4.3}$$

For terrain slopes reaching the 500 mb level the 7090 mountain term is 16 2/3 times as large as in the 704 version.

The friction term is derived as follows. Denote the components of the mass transport due to surface friction by M_x and M_y

$$(4.4) \qquad M_x = \int_0^\infty \rho u \, dz \qquad M_y = \int_0^\infty \rho v \, dz$$

Then the surface stresses τ_x and τ_y are given by

$$(4.5) \qquad \tau_x = f M_y \qquad \tau_y = -f M_x$$

We can now write an expression for the vertical velocity, ω_H, at the top of the friction layer due to mass divergence induced within the layer by surface friction.

$$(4.6) \qquad \begin{aligned} \omega_H &= -\int_{p_g}^{p_H} \nabla \cdot \mathbb{V} \, dp = g\bar{\rho} \int_0^H \nabla \cdot \mathbb{V} \, dz \\ &= g\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}\right) = \frac{g}{f}\left(\frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x}\right) \end{aligned}$$

Or expressing the stress in terms of a drag coefficient, C_D, times the kinetic energy per unit volume.

$$(4.7) \qquad \omega_H = \frac{g\bar{\rho}}{f}\left[\frac{\partial}{\partial y}(C_D u_g V_g) - \frac{\partial}{\partial x}(C_D v_g V_g)\right]$$

where u_g and v_g are the components of the ground level wind of speed V_g. Using the same approach as for the mountain term leads to the following friction term for the barotropic vorticity equation

$$\eta \frac{\partial \omega_H}{\partial p} = \frac{n g \bar{\rho}}{f(P_g - 2\times 10^5)} \left[\frac{\partial}{\partial y}(C_D u_g V_g) - \frac{\partial}{\partial x}(C_D v_g V_g) \right] \tag{4.8}$$

With these changes the complete barotropic forecast equation in the 7090 version of the mesh model is as follows

$$\left(\nabla^2 - \frac{4n}{\bar{\psi}}\right)\frac{\partial \psi}{\partial t} = -1.13\, J(\psi, \eta) + \frac{n r}{(P_g - 2\times 10^5)} J(\psi, P_g) - \frac{g \bar{\rho} n r^2}{f(P_g - 2\times 10^5)} \left\{ \frac{\partial}{\partial x}\left[C_D \frac{\partial \psi}{\partial x} \sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2} \right] + \frac{\partial}{\partial y}\left[C_D \frac{\partial \psi}{\partial y} \sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2} \right] \right\} \tag{4.9}$$

Scaling of the 7090 barotropic model is the same as in the 704 version except for P_g. $\hat{p}$, $\bar{\rho}$ and r are defined as follows

$$P_g = 2\hat{p} \times 10^6$$

$$\bar{\rho} = 1.2 \times 10^{-3}$$

$$r = 1.8 - 3.2\hat{p}$$

Expressing equation (4.9) in finite difference form, multiplying through by 7200 d^2/m^2, scaling one ψ in each term and substituting values leads to equation (2.34) for the left hand side and the vorticity advection term. However, the latter is now multiplied by 1.13 increasing the coefficient to 4.53.

The mountain term becomes

$$\frac{7200 d^2 n r}{m^2(P_g - 2\times10^5)} \times \frac{m^2}{4d^2} J(\hat{\psi}, P_g) = \frac{7200\, n r}{4(P_g - 2\times10^5)} J(\hat{\psi}, P_g)$$

$$= \frac{7200\, r (22.24)(10^{-4}) n}{4\times 2(\hat{p} - .1)10^6} J(\hat{\psi}, 2\hat{p}\times10^6)$$

$$= 4.0032 \frac{r\hat{n}}{(\hat{p} - .1)} J(\hat{\psi}, \hat{p})$$

The friction term becomes

$$-\frac{7200 d^2 n r^2 \bar{\rho} g}{m^2 f (P_g - 2\times10^5)}\left[\frac{m}{2d}\delta_x\left(C_D \frac{m}{2d}\Delta\psi_x \sqrt{\frac{m^2}{4d^2}\left[(\Delta\hat{\psi}_x)^2 + (\Delta\hat{\psi}_y)^2\right]}\right)\right.$$

$$\left. + \frac{m}{2d}\delta_y\left(C_D \frac{m}{2d}\Delta\psi_y \sqrt{\frac{m^2}{4d^2}\left[(\Delta\hat{\psi}_x)^2 + (\Delta\hat{\psi}_y)^2\right]}\right)\right]$$

$$= -\frac{4.0032\, m \bar{\rho} g r^2 \hat{n}}{16(\hat{p} - .1)10^6 d f} 2^{17} \frac{g}{f}\left[\delta_x\left(C_D \Delta\hat{\psi}_x \sqrt{(\Delta\hat{\psi}_x)^2 + (\Delta\hat{\psi}_y)^2}\right)\right.$$

$$\left. + \delta_y\left(C_D \Delta\hat{\psi}_y \sqrt{(\Delta\hat{\psi}_x)^2 + (\Delta\hat{\psi}_y)^2}\right)\right]$$

$$= -\frac{4(66.0169) r^2 \hat{n}}{\hat{p} - .1}\left(\frac{1 + \sin 60}{1 + \sin\phi}\right)\frac{1}{\sin\phi}\left[\qquad\right]$$

Combining these we obtain the barotropic forecast equation solved by the 7090

$$\left(\nabla^2 - .942\frac{\hat{n}}{\hat{p}^2}\right)\Delta_t\hat{\psi} = -4.52\,J(\hat{\psi},\hat{n})$$

$$+\frac{4r\hat{n}}{(\hat{p}-.1)}\left\{J(\hat{\psi},\hat{p}) - 66\,r\left[\frac{1+\sin 66}{(1+\sin\phi)\sin\phi}\right]\times\right.$$

$$\left.\times\left[\delta_x\left(C_D\,\Delta_x\hat{\psi}\sqrt{(\Delta_x\hat{\psi})^2+(\Delta_y\hat{\psi})^2}\right)+\delta_y\left(C_D\,\Delta_y\hat{\psi}\sqrt{(\Delta_x\hat{\psi})^2+(\Delta_y\hat{\psi})^2}\right)\right]\right\}$$

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SUPPLEMENT TO JNWP OFFICE NOTE NO. 15 (revised 1 Oct. 1960)

ON THE DERIVATION OF THE OPERATIONAL BAROTROPIC SCALED FINITE-DIFFERENCE EQUATIONS

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The purpose of this supplement to JNWP Office Note No. 15 (Revised 1 October 1960) is to provide a more logical basis for the derivation of the scaled finite-difference barotropic equations. This supplement covers the same material as on pp. 11-13.

The derivation proceeds from equation (2.30) on page 9, and ends with equation (2.34) on page 13.

Equation (2.30):

$$\left(\nabla^2 - \frac{\mu}{\psi}\eta\right)\frac{\partial \psi}{\partial t} = -J(\psi, \eta) + a\eta J\left(\psi, \frac{p_g}{p_0}\right)$$

Let us deal also with the evaluation of vorticity. To this end we insert an equation (2.30a):

Equation (2.30a):

$$\eta = \frac{g}{f}\nabla^2\psi + f$$

SCALING AND FINITE DIFFERENCES

$$\hat{\psi} = \frac{\psi}{g\ell/\bar{f}}$$

$$\hat{f} = \frac{f}{8/\Delta t}$$

$$\hat{\eta} = \frac{\eta}{8/\Delta t}$$

$$\hat{M}^2 = \frac{m^2}{\bar{m}^2}$$

$$\bar{m}^2 = \frac{8\bar{f}d^2}{g\ell\cdot\Delta t}$$

$$\hat{p} = \frac{p_g}{2.5\,p_0}$$

$$\hat{\bar{\psi}} = \frac{\bar{\psi}}{g\ell/\bar{f}}$$

$$\frac{\partial}{\partial t} \cong \frac{\Delta_\tau}{2\cdot\Delta t}$$

$$\nabla^2 \cong \frac{m^2}{d^2}\,\mathbb{V}^2$$

$$J \cong \frac{m^2}{4d^2}\,\mathbb{J}$$

$$\Delta_r \psi = \psi_{r+1} - \psi_{r-1}$$

$$\nabla^2 \psi = \psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j}$$

$$\mathcal{J}(\psi, \eta) = (\psi_{i+1,j} - \psi_{i-1,j})(\eta_{i,j+1} - \eta_{i,j-1}) - (\psi_{i,j+1} - \psi_{i,j-1})(\eta_{i+1,j} - \eta_{i-1,j})$$

Substitution into equation (2.30) and (2.30a) yields equation (2.34), on page 13, and equation (2.34a) which we here insert:

Equation (2.34):

$$\left(\nabla^2 - \frac{\mu}{\hat{\psi}} \frac{\hat{\eta}}{\hat{M}^2}\right) \Delta_r \hat{\psi} = 4\,\mathcal{J}(\hat{\eta}, \hat{\psi}) + 10\text{-}a\,\hat{\eta}\,\mathcal{J}(\hat{\psi}, \hat{p})$$

Equation (2.34a):

$$\hat{\eta} = \hat{M}^2 \left(\nabla^2 \hat{\psi} + \frac{\hat{f}}{\hat{M}^2}\right)$$

SCALING PARAMETERS

$$g = 980 \text{ cm sec}^{-2}$$
$$\ell = 2^{17} \text{ cm}$$
$$\bar{f} = 2\Omega \sin 45^\circ$$
$$\Omega = 0.72921 \times 10^{-4} \text{ sec}^{-1}$$
$$\Delta t = 1 \text{ hr}$$
$$d = 381 \text{ km}$$
$$p_0 = 1000 \text{ mb}$$
$$a = 0.2$$
$$\mu = 4$$
$$\bar{\Psi} = \frac{gH}{\bar{f}}$$
$$H = 18{,}281 \text{ ft}$$

Note that the derviation of equations (2.34) and (2.34a) is independent of the values of the scaling parameters.